A Theoretical Demonstration of the Normal Stress Formula in the Hook

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ABSTRACT
This paper describes is in a concise and clear way the revisiting of a classic element of the mechanics which is the hook .To carry out the theoretical formulation, the classical hypotheses of construction science have been used, in particular : Bernoulli hypothesis for plane sections under load, curved beam under load, normal stress formulation and Hooke's law.

Keyword
Construction Science, Normal Stress, Curved Beams, Hook, Neutral Axis.

1. INTRODUCTION
Let's start our work considering the picture below where a classic hook have been drawn. The hook is loaded with a vertical force P. Stresses in section I - II are defined in a simple manner using the equations of bending of right beams or more accurately considering the neutral axis curvature of the hook. In the more general case, at every curve section of the hook there act three force factors : a Normal Force, a Shear Force and a Bending Moment.

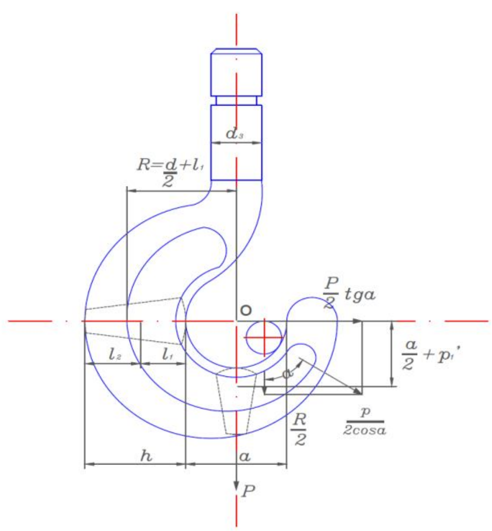

Fig. 1. A classical Hook

1.1 Method
In order to proceed with analysis, we will simplify the geometry of the hook, considering only the lower part of the element which undergoes to the highest stress.

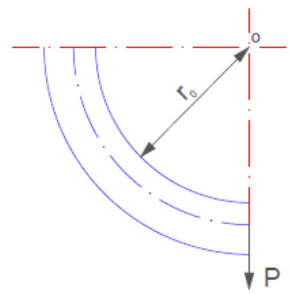

Fig. 2 Semplified representation

Let's proceed considering the beam as wedged to one side and loaded with the force P to the other, Figure 3,, furthermore let's cut this element with a plane, named m-n and get rid of the right part. In order to maintain the equilibrium of forces and moments, we should apply to the remaining part a force P and a moment.

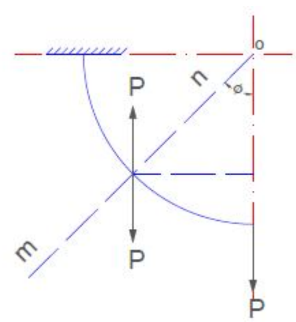

Fig. 3. Force acting on the element

Let's decompose the force P in two components, a normal, N, and a tangential one, Q, refered to the normal section, figure 4.

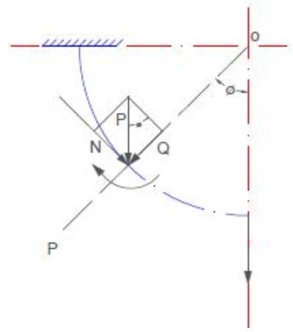

Fig. 4. Forces acting on the element

The sign of the forces and moment remain the same as in the case of a straight beam, exception makes the bending moment M which is accepted positive when it increases the bending.
In general we will have $M = -P \times r_0 \times \sin\varphi$

$Q = P\cos\varphi$, $N = P\sin\varphi$

We can see that we fall in the case of the compound resistance of the beam under traction (N) and transversal bending (M, Q). In comparison with the straight beam case, the difference will appear in the deformation and will be caused by the fact that in the curved beam, the length of two points between two adjacent cross sections is different while in the straight beam is the same. The impact of the shear force will not be considered upon the strain of the bended beam due to the fact that it's impact is negligible and the tangential stress can be calculated the same way as in the straight beams according to the formula

$$\tau = \frac{Q * S_y}{I_y * b}$$

1.2 The strain

We will consider the bending of the beam that falls under the clear bending situation, neglecting the shear force Q. Let's refer to the following section, cutted with a plane named c-d and subjected to the moment M and to the normal stress as showed in figure 5 and 6

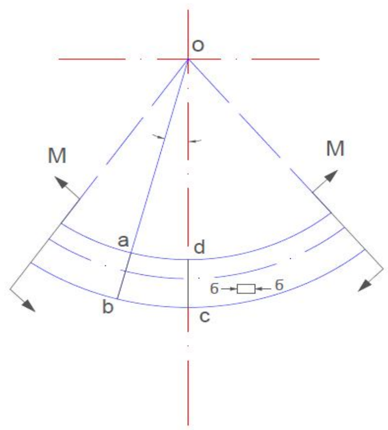

Fig. 5

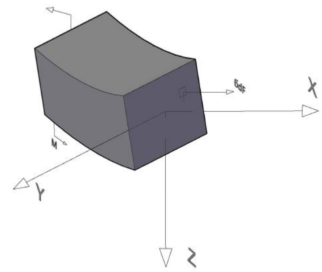

Fig. 6. Infinitely small element extracted

The equations that express the equilibrium of the section can be written as follows:

$$\sum Nx = \int_A \sigma dA = 0$$

$$\sum M_y = \int_A \sigma * z * dA = 0$$

$$\sum M_z = \int_A \sigma * y * dA = 0$$

The above equations are insufficient to define σ because of the fact that we don't know the law of distribution of σ in the cross section of the beam. In fact, to around this obstacle, we will use the findings and conclusions of the theory of elasticity who states that sections remains perpendicular to the axis, furthermore, longitudinal curved fibers, during bending will be accepted to not suppress each other.
Given these concessions, we will study the deformation of a curved beam element that arises under the influence of normal force and bending moment, we will apply here the principle of independence. Normal force N , being central , creates distortions uniform over all the section , therefore the strains will be uniform and will result :

$$\sigma = \frac{N}{A}$$

1.3 Bending Moment

Let us now analyze the effect of the bending moment over the deformation of the curved beam, in order to do so we must detach the curved element a-b-c-d through the same planes passing through the centre of curvature. The left section a-b of the element a-b-c-d will be held stationery with respect to the right part and rotated around the neutral axis with the $\Delta d\emptyset$ angle, as showed in figure 7

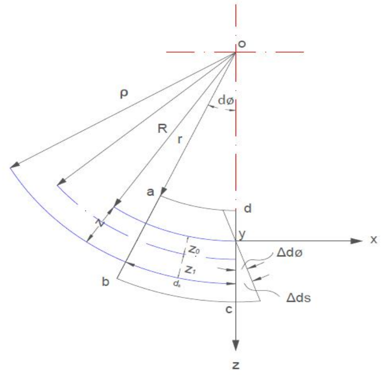

Fig. 7

Fibers distancing z from neutral layer and with initial length

$d_s = (r + v) * \Delta d\emptyset$ will undergo extensions $\Delta d_s = z * \Delta d\emptyset$

where extension of each point relative to the height z from the neutral axis will be

$$\varepsilon = \frac{\Delta d_s}{d_s} = \frac{z}{r + z} * \frac{\Delta d\emptyset}{d\emptyset}$$

r = curvature radius of the neutral axis.

The abovementioned equation, derived from reasoning of remaining plane of the cross sections, gives the hyperbolic law of change of relative deformations to the longitudinal points with respect of the height z.
During the pure bending case, we will accept the release that longitudinal fibers do not exchange forces between them, according to this release, the strain state will be linear and can be expressed from Hooke's law:

$$\sigma = \varepsilon * E = E * \frac{\Delta d\emptyset}{d\emptyset} * \frac{z}{r + z}$$

In this way, normal stress, as well as relative extensions are distributed over the section according to the hyperbolic law. The asymptote of the hyperbole is a normal line to z axis and distanced z = -r from neutral axis y, showed in figure 8.

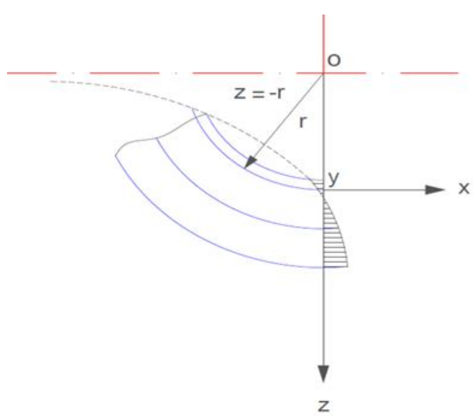

Fig. 8. Hyperbolic distribution

From the first equation of equilibrium system is defined the curvature radius of the neutral layer and the shift z_0 of neutral axis from the center of gravity of section .

Replacing the value of σ in the first equation will receive :

$$\int_A \boldsymbol{\sigma dA} = \boldsymbol{E} * \frac{\boldsymbol{\Delta d\theta}}{\boldsymbol{d\theta}} * \int_A \frac{\boldsymbol{z}}{\boldsymbol{r + z}} \boldsymbol{dA} = \boldsymbol{0}$$

Since $E * \frac{\Delta d\theta}{d\theta} \neq 0$ then $\int_A \frac{z}{r+z} dA = 0$
From picture : $r + z = \rho$ and $z = \rho - r$
Where : ρ is curvature radius of a fixed line distanced z from neutral layer
$\int_A \frac{z}{r+z} * dA = \int_A \frac{\rho - r}{r+z} dA = \int_A \frac{\rho}{r+z} dA - r * \int_A \frac{dA}{r+z} = \int_A dA - r \int_A \frac{dA}{\rho} = 0$
From this equation we define the curvature radius of neutral layer

$$r = \frac{A}{\int_A \frac{dA}{\rho}}$$

The distance z_0 of the gravity centre of the section from neutral axis will be :

$$z_0 = R - r = R - \frac{A}{\int_A \frac{dA}{\rho}}$$

By substituting σ in the second equation of equilibrium we get :

$$M = \int_A \sigma * z * dA = E \frac{\Delta d\emptyset}{d\emptyset} \int_A \frac{z^2}{r + z} dA$$

Expression : $\int_A \frac{z^2}{r+z} dA$ gives the static moment on the y axis

$\int_A \frac{z^2}{r+z} * dA = \int_A \frac{z^2 + r*z - r*z}{r+z} * dA = \int_F \frac{z*(r+z)}{r+z} dF - r \int_F \frac{z}{r+z} dF = \int_F z * dF - 0 = S_y$

According to results we can derive that

$$\frac{\Delta d\emptyset}{d\emptyset} = \frac{M}{E * Sy}$$

Finally we can get the expression of normal stress due to the bending moment

$$\sigma = \frac{M}{S_y} * \frac{z}{r + z}$$

The third equilibrium equation can be written as follows

$$\int_F \sigma * y * dF = E * \frac{\Delta d\emptyset}{d\emptyset} * \int_F \frac{y * z}{r + z} * dF = 0$$

Consequently $\int_A \frac{y*z}{r+z} * dA = 0$

This ending expresses the fact that cross section is symmetrical to the z axis.

From expression $S_y = \int_A \frac{z^2}{r+z} dA$ can be defined that $Sy > 0$ always. On the other hand $S_y = A * z_0 > 0$. Since A is positive, z_0 will be positive to, so, during the bending of the curved beam the neutral layer and y axis will always be displaced toward the centre of curvature compared to the section's center of gravity.

2. ALTERNATIVE METHOD

The theory of pure bending of the curved beam is a technical and approximated theory since it neglects the impact of reciprocated action between material fiber, however it has a suficient precision for practical calculations.

Based on this theory and reasoning for calculation of hook which is one of the most classic cases of curved beam subject to bending, can be derived another formulation that we are going to explain subsequently.

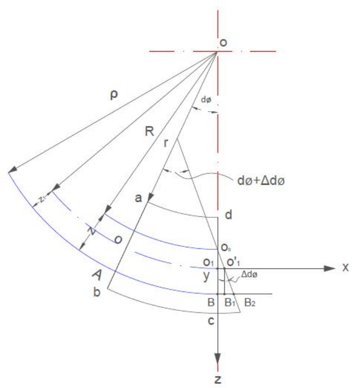

Fig. 9. Relative longitudinal deformation

$$\varepsilon = \frac{BB_1}{AB} = \frac{BB_2 + B_2B_1}{AB}$$

According to figure 9, $BB_1 = O_1O_1^{'}$ express relative extension of point $O_1O_1^{'}$.

$$\varepsilon_0 = \frac{O_1O_1^{'}}{OO_1} \rightarrow O_1O_1^{'} = \varepsilon_0 * OO_1$$

$$BB_2 = O_1O_1^{'} = \varepsilon * R * d\theta$$

$$B_2B_1 = z_1 * d\theta$$

Replacing we get :

$$\varepsilon = \frac{\varepsilon_0 * R * d\theta + z_1 * \Delta d\theta}{(R + z_1) * d\theta} = \frac{\varepsilon_0 * R + z_1 * \frac{\Delta d\theta}{d\theta}}{R + z_1}$$

Subsequently we add and subtract $\varepsilon_0 * z_1$

$$\varepsilon = \frac{\varepsilon_0 * R + \varepsilon_0 * z_1 - \varepsilon_0 * z_1 + z_1 * \frac{\Delta d\theta}{d\theta}}{R + z_1} = \varepsilon_0 + \left(\frac{\Delta d\theta}{d\theta} - \varepsilon_0\right) * \frac{z_1}{R + z_1}$$

Remembering the simplifications made on the first case we can apply the Hooke's law

$$\sigma = E * \varepsilon = E * \left[\varepsilon_0 + \left(\frac{\Delta d\theta}{d\theta} - \varepsilon_0\right) * \frac{z_1}{R + z_1}\right]$$

We should substitute this expression in the second equilibrium equation to obtain the bending moment.

$$M = \int_A \sigma * z_1 dA = \int_A E * z_1 * \left[\varepsilon_0 + \left(\frac{\Delta d\theta}{d\theta} - \varepsilon_0\right) * \frac{z_1}{R + z_1}\right] * dA$$

$$M = E * \varepsilon_0 \int_A z_1 dA + E * \left(\frac{\Delta d\theta}{d\theta} - \varepsilon_0\right) * \int_A \frac{z_1^2}{R + z_1} * dA$$

The first part of the integral is null because it expresses the static moment of resistance on the y axis, passing on the center of gravity of the section, so it remains only the second part

$$M = E * \left(\frac{\Delta d\theta}{d\theta} - \varepsilon_0\right) * \int_A \frac{z_1^2}{R + z_1} * dA$$

Remembering the first equilibrium equation $\int_A \sigma\, dA = 0$, we can write:

$$\int_A E * \left[\varepsilon_0 + \left(\frac{\Delta d\theta}{d\theta} - \varepsilon_0\right) * \frac{z_1}{R + z_1}\right] * dA = 0$$

whence :

$$\varepsilon_0 * \int_A dA + \left(\frac{\Delta d\theta}{d\theta} - \varepsilon_0\right) * \int_A \frac{z_1}{R + z_1} * dA = 0$$

$$\varepsilon_0 * A = -\left(\frac{\Delta d\theta}{d\theta} - \varepsilon_0\right) * \int_A \frac{z_1}{R + z_1} * dA = 0$$

If we remark $K = -\frac{1}{A}\int_F \frac{z_1}{R + z_1} * dA$, we will have

$$\varepsilon_0 = \left(\frac{\Delta d\theta}{d\theta} - \varepsilon_0\right) * K \rightarrow \left(\frac{\Delta d\theta}{d\theta} - \varepsilon_0\right) = \frac{\varepsilon_0}{K}$$

Let us consider now the integral

$$\int_A \frac{z_1^2}{R + z_1} * dA = \int_A \frac{z_1^{'} + z_1 * R - z_1 * R}{R + z_1} * dA = \int_A \left(z_1 - R * \frac{z_1}{R + z_1}\right) dA =$$
$$\int_A z_1 dF - R * \int_A \frac{z_1}{R + z_1} * dA = -R \int_A \frac{z_1}{R + z_1} * dA$$

For as much as the first term is null, let's consider the second term by multiplying and dividing for the area A

$$-R * \frac{A}{A} \int_A \frac{z_1}{R + z_1} * dA = R * A * K$$

Substituting in the moment's expression we get :

$$M = E * \left(\frac{\Delta d\theta}{d\theta} - \varepsilon_0\right) * K * A * R = E * \frac{\varepsilon_0}{K} * K * A * R = E * \varepsilon_0 * A * R$$

Then : $\varepsilon_0 = \frac{M}{E * A * R}$ and $\left(\frac{\Delta d\theta}{d\theta} - \varepsilon_0\right) = \frac{M}{E * K * A * R}$

Replacing in the equation of normal stress we can write :

$$\sigma = E * \left[\varepsilon_0 + \left(\frac{\Delta d\theta}{d\theta} - \varepsilon_0\right) * \frac{z_1}{R + z_1}\right] = E * \left[\frac{M}{E * A * R} + \frac{M}{E * A * K * R} * \frac{z_1}{R + z_1}\right]$$

Finally

$$\sigma = \frac{M}{A * R} + \frac{M}{K * A * R} * \frac{z_1}{R + z_1}$$

Where : $R \rightarrow$ curvature radius of the section's gravity centers.
$z_1 \rightarrow$ distance of the studied layer from this axis.

In the general case of plane bending of the beam with curved axes in it's section, except the bending moment we will have also the normal force N and the shear force Q.
The last one in not influent and we usually neglect it, so the normal stress generated in the section for this case can be defined with the formula :

According to the first aproach

$$\sigma = \frac{N}{A} + \frac{M}{S_y} * \frac{z}{r + z}$$

According to the second aproach

$$\sigma = \frac{N}{A} + \frac{M}{A * R} + \frac{M}{K * A * R} * \frac{z_1}{R + z_1}$$

Both formulas give the same result. The difference lays on the fact that in the first approach the impact of the section and the curvature is given through the curvature radius of the neutral axis, given by the formula $r = \frac{A}{\int_A \frac{dA}{\rho}}$
Meanwhile in the second aproach through the coefficient k which is numerical and given by the formula:

$$K = -\frac{1}{A}\int_A \frac{z_1}{R + z_1} * dA$$

The second formulation is mostly used for engineering calculations, the difficulty lays mostly on the definition of the same coefficient k because the integral must be solved.
For the solution of the integral is necessary for any given interval, the recognition of the dependence function between the variables that are under the sign of the integral, depending on which is determined in accordance with the shape and size of the cross section of the beam .
For simple geometric figures, calculating the coefficient k can be performed under different accounting formulas , considering the distance of the centre of curvature on the axis of the hook
(Where in fact the improvement aims)

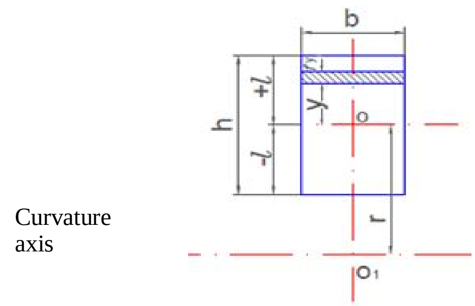

Fig. 10. Rectangular section

For rectangular rectangle, figure 10, and for the ratio $e \div r = u$ and

$\ln x = 2.3026 \ln y$, $K = -1 + 0.5 * u * \ln \frac{1+u}{1-u}$

If $\omega < 1$ then $K = \frac{1}{3} * \omega^2 + \frac{1}{5} * \omega^4 + \frac{1}{7} * \omega^6$

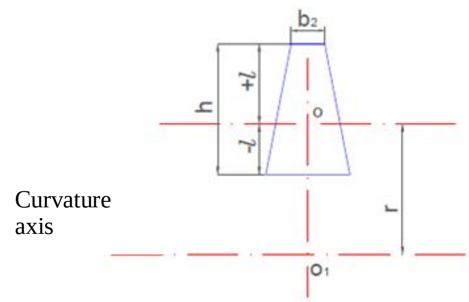

Fig. 10. Trapeziodal section

For the trapeze,in figure 11, for the sides ratio $b_1 : b_2 = m$

$$K = -1 + \frac{2}{m + 1} * \frac{r}{h}\left\{\left[1 + (m - 1) * \frac{r + b_2}{r + b_1} - (n - 1)\right]\right\}$$

For triangle (particular case of the trapeze where b1=0)

$$K = -1 + 2 * \frac{r}{h}\left\{\left[\frac{2}{3} + \frac{r}{h}\right] \ln \frac{3 * r + 2 * h}{3K - h}\right\}$$

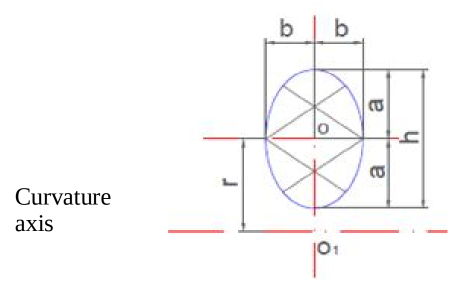

Fig.12. Elliptical section

For the circle or the ellipse, figure 12,whom longest half-axis lays on the plane of the bending of the beam

$$K = \frac{1}{4} * \omega^2 + \frac{1}{8} * \omega^4 + \frac{1}{64} * \omega^6$$

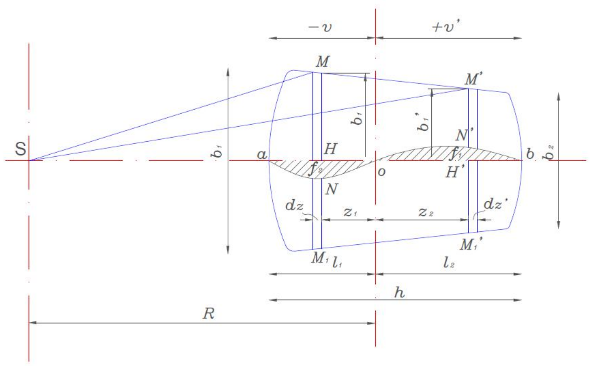

Fig. 11. Determination of coefficient k

Where **w** is taken in accordance with R/r$\frac{R}{r}$ or a/r ratios

Sometimes, k coefficient can also determined graphically.

The graphic drawing is made (for the example showed in figure 13 and for many others) by connecting e.g. point M (or M') in the contour of the cross section (trapeze in our example, which is the most acceptable in the case of the hook) with the centre of curvature S.

From the centre of gravity of the section, O, we let pass the straight line ON (or ON') parallel with line SM (or SM_1) until the intersection with the horizontal MM_1 (or $M'M'_1$) in the point N (or N')

Repeating the same procedure for all the vertical lines, we get a series of points, connecting them with continuous spline gives us the two surfaces f_1 and f_2 who met together in the O point.

From triangles similarity, SMH and ONH, we will have :

$$\frac{NH}{MH} = \frac{z_1}{K + z_1}$$

In this equality we must consider the sign of z1, positive on the right and negative on the left of the centre of gravity (in the formula the sign is not reflected)

Thus, considering the sign, coefficient K can be writen :

$$K = -\frac{1}{A}\int_A \frac{NH}{MH} * dA$$

From figure we can write that:

$$dA = 2 * M * H * d_z$$

And replacing:

$$dA = -\frac{2}{A}\int_{-v}^{+v} NH * d_z$$

From where we obtain

$$K = -\frac{2}{F}(f_2 - f_1)$$

Or

$$K = \frac{2}{F}(f_2 - f_1)$$

3. CONCLUSIONS

Normal stress definition and formulation is one of the most important study parts of the construction science subject, both in university and in real practice applications. As a particular case, the curved beams, such as the hook, due to the curved shape are double stressed because even if aren't loaded with normal forces, they are subjected to normal stresses, increased furthermore due to the born of bending moment on the section. This combination of variables drove us to perform the analysis. The paper compares two methods for determining the normal stress into the curved beam who works in bending. In the work have been processed and adapted the formulas of the straight beams who works in bending, in the case of the curved beams.

On the basis of studies of the theory of elasticity and results obtained experimentally, we studied the deformations of the sections of the curved beam on the basis of the principle of independence of application of the forces. As a consequence of the hypothesis that perpendicular sections remain flat and the hyperbolic law of the relative deformations of the longitudinal fibers in the section of the curved beam , we obtained the calculation formulas where the hook is the most evident case of the curved beam where these formulas can be applied.

In the first method, the section's curvature radius, calculated on the neutral axis, is related to the bending moment and though, to the normal stress. In the second method, the section's curvature radius is further elaborated and defines a numerical factor K, more suitable for practical applications particularly in simple and commonly used sections of beams. The advantage of the second method lies on the possibility to define numerically the coefficient K.

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