Numerical Solution for Semi Linear Hyperbolic Differential Equations

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ABSTRACT
A Numerical method is proposed for the solution of semi linear hyperbolic differential equation. The method is of the characteristic type and it is particularly helpful in case of discontinuous initial data. The new proposal is elaborated and implemented in Matlab for canonical case of differential equation. Then, its local and global order of convergence is conventionally estimated using canonical examples. The method is extended for the solution of a wave equation with a non-classical right boundary condition, where special treatment of the right boundary is required.

Keywords
Characteristic Method, Conventional Order, Boundary Conditions, Semi Linear Hyperbolic, Linear Hyperbolic.

1. INTRODUCTION
Several applications in engineering, particularly in solid mechanics, involve hyperbolic differential equations that have discontinuous initial data. In such problems, finite difference methods fail to be used because these methods tend to smooth out discontinuities. The method of characteristics can be applied for these cases and formally it can be developed for the general quasi linear hyperbolic equation of the form

$$a_1 u_{tt} + a_2 u_{tx} + a_3 u_{xx} = a_4, \quad 0 < x < l, \quad t > 0, \qquad (1)$$

where $a_i = a_i(x, t, u, u_x, u_t)$, for $i = 1, 2, 3, 4$, denotes a nonlinear function.

The initial conditions are

$$u(x, 0) = \varphi(x), \quad 0 \le x \le l, \qquad (2)$$

$$u_t(x, 0) = \psi(x), \quad 0 \le x \le l, \qquad (3)$$

meantime, different type of boundary conditions are applied in certain cases.

The characteristics curves, which are the solutions of the odes

$$\frac{dt}{dx} = \frac{a_2 \pm \sqrt{a_2^2 - 4a_1 a_3}}{2a_3} \qquad (4)$$

cannot be determined nether analytically, not numerically, because the right-hand functions in (4) depend formally on x, t, u, u_x, and u_t. Numerical methods that use characteristic curves encounter innumerous difficulties when applied in practice and they generally are not recommended in numerical literature [3]. A combination of the method of characteristics with Richardson extrapolation is used successfully in [2], for a special nonlinear wave equation, derived and analyzed analytically in [1].

In this paper we explore in details and formalize the idea given in [2] for the semi linear case of equation (1), when the function $a_i = a_i(x, t)$, for $i = 1, 2, 3$, and $a_4 = a_4(x, t, u, u_x, u_t)$. In this case the odes (4) are "normal" and, using for example some useful Matlab utilities, their characteristics curves, could be efficiently computed and also drawn in a space of variables x and t.

The idea of the new proposal is presented in the next section and two basic related questions are raised there. In section III we elaborate the method and implement it in case of canonical form of equation (1). Using specific examples with known exact solutions, the local and global conventional orders of accuracy are estimated for the new method. It is found that these orders are respectively 1 unit higher, compared to those of classic characteristic method. In section IV it is told, how the non-classical boundary conditions of nonlinear equation in [1, 2], can be implemented in the context of the new proposal.

2. NUMERICAL SOLUTION FOR THE SEMI LINEAR EQUATION (1-3)
A mesh G_0 is obtained by discretizing the interval $0 \le x \le l$ into m subintervals each of width $h = l/m$. Denote by [0, l] x [0, T] the physical space of variables x and t. In Fig. 1, the characteristic curves that pass through the points of G_0 are drawn (bold solid lines). In the "triangle" region OLM, the solution of (1) is influenced (and so it is determined), only by the initial conditions (2)-(3).

It is assumed that $\phi'(x)$ exists so that $u_x'(x, 0) = \phi'(x)$, whenever $0 \le x \le l$. Consequently, $u(x, t)$, $u_x'(x, t)$ and $u_t(x, t)$ will be known functions whenever $0 \le x \le l$ and $t = 0$. Suppose $P(x_1, t_1)$, $Q(x_2, t_2)$ and $S(x_3, t_3)$ with $t_1 = t_2 = t_3 = 0$, are three successive points along x-axis so that $0 < x_1 < x_2 < x_3 < 1$. Denote by (P, Q, R) the standard process of classic characteristic method for points P, Q and R. If the processes (P, Q, R), (Q, S, T), (R, T, V) are applied repeatedly as it is shown in Fig. 1, then approximations for $u_x'(V)$, $u_t'(V)$ and $u(V)$ can be received in two ways:

1) One way is to apply the single process (P, S, V) with step size 2h. We can denote the approximations received as $u_x'(V, 2h)$, $u_t'(V, 2h)$ and $u(V, 2h)$.

2) The second way is to apply the triple process (P, Q, R), (Q, S, T), (R, T, V). We can denote these approximations as $u_x'(V, h)$, $u_t'(V, h)$ and $u(V, h)$.
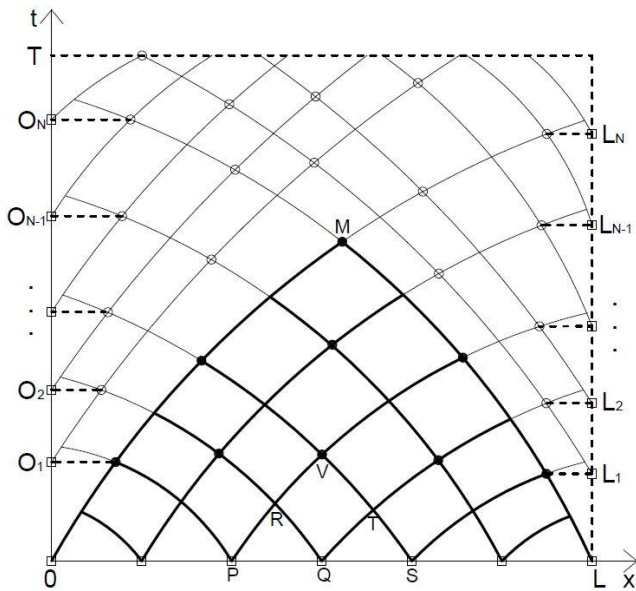
Figure 1: Example of mesh where the solution of (1-3) is computed.

We can follow in that way inside the region OLM. We can extend this process N times (N depended on T) outside the region OLM, as in Fig. 1, considering and using specific boundary conditions imposed on equation (1). Clearly, a mesh G by (m+1) x (N+1) points is created, and in each interior point of this mesh, the solution is computed in ways 1) and 2) as above. The balls in Fig. 1, filled or empty, are the interior nodes of G. The symbol □ is used for the initial or boundary nodes. Using properly useful Matlab utilities, *ode45* and *polyxpoly*, the mesh G could be primarily computed and the coordinates x, and t of points of G could be respectively allocated in two matrices A_x and A_t with the same dimensions (m+1) x (N+1). Meantime, we can see in Fig.1 that another mesh G' and two other matrices B_x and B_t, with dimension (m-1) x N are configured, where the solution is computed by the simple process (P, Q, R). The nodes of G' are all intersections of the characteristics curves that are not balled. So, the processes (P, Q, R), (Q, S, T), (R, T, V), and (P, S, V) above are carried out moving through the matrices A_x, A_t, B_x and B_t.

Two basic questions arise here:

a) Could the two different approximations in points 1) and 2) above, be used to produce better approximations?

c) Could the accuracy expected to be attained at point a) above, be maintained for the approximations done at the boundary nodes?

The answers for these two questions and related matters will be subject of next sections. To get some insight we will consider the canonical form of equation (1)

$$u_{tt} - u_{xx} = f(x, t, u, u_x, u_t) \tag{5}$$

with initial conditions.

$$u(x, 0) = \varphi(x), \quad 0 \le x \le l, \tag{6}$$

$$u_t(x, 0) = \psi(x), \quad 0 \le x \le l, \tag{7}$$

Really, the semi linear equation (1) can be transformed into canonical form (5) by the successive variable transformations

$$\begin{cases} \xi = \phi_1(x, t) \\ \eta = \phi_2(x, t) \end{cases} \text{and} \begin{cases} t = \xi + \eta \\ x = \xi - \eta \end{cases}$$

$\phi_1(x, t) = c_1$ and $\phi_2(x, t) = c_2$ being the analytic solutions of odes (4).

3. NUMERICAL SOLUTION FOR THE PROBLEM (5-7)

It is easy to see that the version of Fig.1, in case of problem (5-7), is Fig. 2 below.
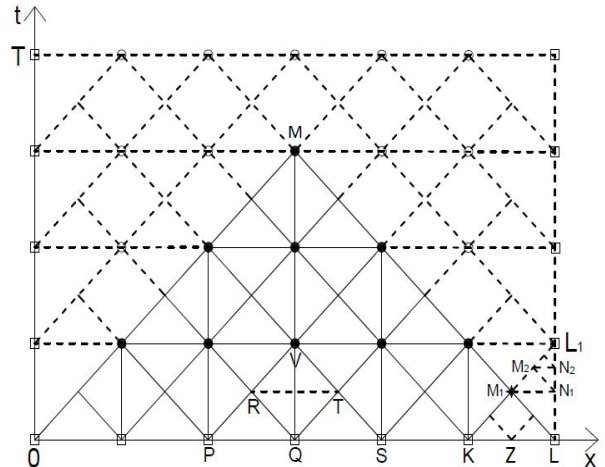

Figure 2: Example of mesh where the solution of (5-7) is computed.

It can be derived that the (P, Q, R) process of the method of characteristics, applied for the problem (5-7) leads to the following system of equations:

$$u_x'(R) = \frac{1}{2}[u_x'(Q) + u_x'(P) + u_t'(Q) - u_t'(P)] + \frac{h}{8}[f(Q) - f(P)], \tag{8}$$

$$u_t'(R) = \frac{1}{2}[u_x'(Q) - u_x'(P) + u_t'(Q) + u_t'(P)] + \frac{h}{8}[f(Q) + 2f(R) + f(P)], \tag{9}$$

$$u(R) = \frac{1}{2}[u(P) + u(Q)] + \frac{h}{8}[u_x'(P) - u_x'(Q) + u_t'(P) + 2u_t'(R) + u_t'(Q)] \tag{10}$$

A Matlab code is written to implement the (P, S, V) process and triple process (P, Q, R), (Q, S, T), (R, T, V), described in part II.

Consider simple Cauchy problem

$$u_{tt} - u_{xx} = 1 \tag{11}$$

with initial conditions

$$u(x, 0) = sin(x) \tag{12}$$

When in equation (5), $f(x, t, u, u_x, u_t) \equiv g(x, t)$, the problem (5-7) is referred as Cauchy problem for the non-homogeneous wave equation, and its analytic solution is

$$u(x, t) = \frac{\varphi(x-t) + \varphi(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi(u)du - \frac{1}{2}\iint_D g(x, t)dxdt \tag{13}$$

where the domain D is the triangle region of determinacy. Applying formula (13) for the problem (11-12), its exact solution is found to be $u(x,t) = \frac{1}{2}[sin(x+t) * sin(x-t)] + xt - \frac{1}{2}t^2$.

Let $Q(x_0, t_0)$ be an arbitrary point with $t_0 > 0$. For n = 1, 2, ..., 10, and $h = 1/2^{n+3}$, consider the triangle PSV with $P(x_0-h, t_0)$, $S(x_0+h, t_0)$, and right angle in $V(x_0, t_0+h)$, as in Fig. 3.
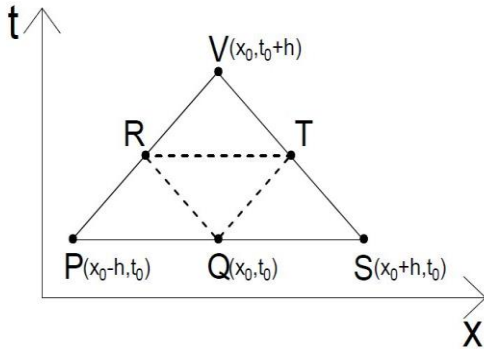

Figure 3: Scheme of nodes for local errors computation.

The problem (11-12) is solved by the method described in parts II and III above, so two approximations of the solutions per each h are obtained at point V. Knowing the exact solution at point V, the local errors of the (P, S, V) process, (column (i) below), and the triple process (P, Q, R), (Q, S, T), (R, T, V), (column (ii)), are computed for 10 different values of step size h. The specific values of h are selected for best performance. Numerical results for the arbitrary point $x_0 = 2$, $t_0 = 3$ are presented in the following table.

Table 1: Local errors for the (P, S, V) process and triple process (P, Q, R), (Q, S, T), (R, T, V).

n	(i)	(ii)
1	3.0840E-07	1.3052E-06
2	3.9673E-08	1.6316E-07
3	5.0291E-09	2.0396E-08
4	6.3300E-10	2.5495E-09
5	7.9399E-11	3.1869E-10
6	9.9420E-12	3.9836E-11
7	1.2444E-12	4.9803E-12
8	1.5554E-13	6.2250E-13
9	1.9540E-14	7.7938E-14
10	2.3315E-15	9.4369E-15

Considering the general expression of local error:

$$l = ch^{r+1}$$

and experimental data of Tab. 1, it is easy to compute the corresponding values c and r per each of the processes above. The corresponding expressions for local errors result to be respectively

$$0.010*h^{2.996} \quad and \quad 0.043*h^{3.002}$$

We suppose that the local order of (P, Q, R) process can be expressed as

$$u(R) - u(R, h) = a_3h^3 + a_4h^4 + ... \approx a_3h^3$$

Therefore

$$u(V) - u(V, 2h) = 8a_3h^3 + 16a_4h^4 + ... \approx 8a_3h^3 = b_3h^3.$$

It is difficult to discuss for the local error at V for the triple process (P, Q, R), (Q, S, T), (R, T, V), symbolically expressed as algebraic summation of three local errors of the form a_3h^3, so its maximum value is expected to be $3a_3h^3$. But it is reasonable to assume that

$$u(V) - u(V, h) = c_3h^3 + c_4h^4 + ... \approx c_3h^3$$

The Matlab commands *corrcoeff* and *polyfit*, with the third and second column of Tab.1 as entries, give respectively the results

```
1.0000E+00      9.9999E-01

9.9999E-01      1.0000E+00

4.2039E+00      -5.8245E-10
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We could read through these results:

1) Strong linear correlation exists between local errors (I) and (II) - as it was expected

2) The proportion b_3/c_3 is ≈ 4.2, which means $c_3 \approx 1.9 \, a_3$ - as it was statistically expected

From this and other experiments done we have concluded that the triple process with step size h has a coefficient of local error about 4 times smaller than the corresponding coefficient of single process with step size 2h. So, the formula

$$u(V, h, 2h) = (4u(V,h) - u(V, 2h)) / 3 \qquad (14)$$

is expected to have a local error of the form d_4h^4. By the formula (14), the order of accuracy is improved by 1 unit. The process of approximating the value of u(V) by the formula (14) will be referred hereafter as (P, Q, S, V) process. In Tab. 2 below are given detailed results for the three processes (methods), for 11 properly selected values of h.

From column (iii) we can find the expression $0.03*h^{4.03}$ for the local error of formula (14), which is in agreement with the comments done above for this formula.

The analysis above for local errors can be extended for the global errors of (P, S, V), triple, and (P, Q, S, V) processes. Let $C(x_0, t_0)$ be an arbitrary point with $t_0 > 1$. Consider the triangle ABC with $A(x_0-1, t_0-1)$, $B(x_0+1, t_0-1)$, and right angle in C, Fig.4. For n = 1, 2, ..., 10, the interval $x_0-1 \le x \le t_0-1$ is discretized in $m = 2^{n+1}$ subintervals each of width $h = 2/m = 1/2^n$.

Table 2: Local errors for (P, S, V), triple, and (P, Q, S, V) processes, at (x0=2, t0=3).

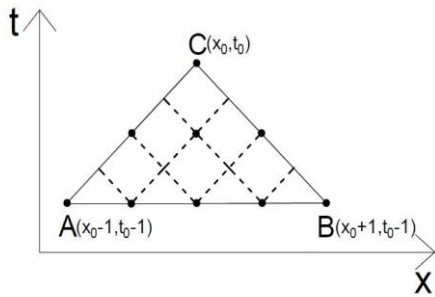| n (h=2^{-n})	(i)	(ii)	iii)
1	1.3431E-04	5.2142E-03	1.5590E-03
2	9.2567E-05	6.6417E-04	9.7965E-05
3	1.6263E-05	8.3411E-05	6.1202E-06
4	2.3231E-06	1.0439E-05	3.8213E-07
5	3.0840E-07	1.3052E-06	2.3867E-08
6	3.9673E-08	1.6316E-07	1.4911E-09
7	5.0291E-09	2.0396E-08	9.3173E-11
8	6.3300E-10	2.5495E-09	5.8226E-12
9	7.9399E-11	3.1869E-10	3.6326E-13
10	9.9420E-12	3.9836E-11	2.2538E-14
11	1.2444E-12	4.9803E-12	8.8818E-16

Figure 4: Example of mesh for global errors computation:

Case: n = 1,   m = 2^{n+1} = 4, and h = 1/2.

The problem (11-12) is solved by the method described in this section, in the triangle ABC, for 11 values of n. So, three approximations of the solution are obtained at point C, per each n. Three corresponding global errors at point C, and respective global error expressions, are computed and displayed in Tab. 3.

Table 3: Global errors and respective errors expressions for (P, S, V), triple and (P, Q, S, V) processes, at (x0=2, t0=3).

n (h=2^{-n})	$0.02*h^{1.96}$	$0.06*h^{1.89}$	$0.02*h^{2.98}$
1	4.1635E-03	8.3851E-03	2.7563E-03
2	1.3095E-03	4.1635E-03	3.5817E-04
3	3.6224E-04	1.3095E-03	4.6484E-05
4	9.5020E-05	3.6224E-04	5.9470E-06
5	2.4320E-05	9.5020E-05	7.5290E-07
6	6.1510E-06	2.4320E-05	9.4739E-08
7	1.5467E-06	6.1510E-06	1.1883E-08
8	3.8778E-07	1.5467E-06	1.4879E-09
9	9.7085E-08	3.8778E-07	1.8615E-10
10	2.4289E-08	9.7085E-08	2.3277E-11
11	6.0743E-09	2.4289E-08	2.9131E-12

The above results indicate that the behavior of global errors is in the same line with that of local errors. As in local case, the global accuracy of (P, Q, S, V) process is 1 unit more than that of (P, S, V) process.

4. NUMERICAL SOLUTION: CASE OF AN INITIAL-BOUNDARY VALUE PROBLEM

By the method described in sections II and III, the numerical solution is developed inside the triangle region of determinacy OLM at Fig. 1 or Fig.2. The extension of the solution in the complementary part of the physical region [0, 1] x [0, T] of variables x and t, is depended on specific boundary conditions that can be imposed on equation (1) or (5). A special case is considered below.

The following model is analyzed in [1], for the vibrations of a string which is fixed at x = 0 and is attached to a dashpot system at x = π: Find the function u(x, t) which satisfies the equation

$$u''_{tt} - u''_{xx} = \varepsilon(u'_t - \tfrac{1}{3}u'^3_t), \quad 0 < x < \pi, \ t > 0, \qquad (15)$$

subject to boundary conditions

$$u(0, t) = 0, \qquad t \geq 0, \qquad (16)$$

$$u'_t(\pi, t) = -\varepsilon\alpha u'_x(\pi, t), \quad t \geq 0, \qquad (17)$$

and initial conditions

$$u(x, 0) = \phi(x), \quad 0 \leq x \leq \pi, \qquad (18)$$

$$u'_t(x, 0) = \psi(x), \quad 0 < x < \pi. \qquad (19)$$

The functions $\phi(x) \ and \ \psi(x)$ above are the initial displacement and the initial velocity of the string, the damping parameter α is a positive constant, and ε is a small dimensionless parameter $(0 < \varepsilon \ll 1)$.

Thus, in (15-19) we have an initial-boundary value problem for a weakly nonlinear partial differential equation with a non-classical right boundary condition.

The numerical method described in this paper can be adopted and applied for the solution of problem (15-19). The classical left boundary condition (16) could be easily implemented in the solution process, and in the way that the accuracy provided by the (P, Q, S, V) process could be maintained at the boundary left nodes. We will treat here, how the non-classical right boundary condition (12) could be implemented. The idea is given in right-down corner of Fig. 2. By the (P, Q, S, V) process applied we can obtain only a local accuracy of order 2 at point M_1 of Fig.2. In order to receive an accuracy of order 3 at this point, the same as in all balled points, we can apply the single process (K, Z, L, M_1). Then we can go on some steps with simple processes (M_1, N_1, M_2), (M_2, N_2, M_3), ..., of type (P, Q, R), which are expected to have good accuracy because of the small step sizes applied. We finalize with good approximations at the point L1. The process is repeated similarly moving farther up the time axis. More details for model (15-19), its full implementation and some numerical results will be reported by the present authors in a coming paper.

5. CONCLUSIONS

A numerical method of the characteristic type is proposed and developed for the solution of semi linear hyperbolic differential equation. The method is implemented for canonical case of equation and its local and global orders of convergence are conventionally estimated. It is found that the orders of convergence are improved by one unit, compared to those of classic method of characteristics. The proposed method can be extended for initial-boundary value problems. An example of a wave equation with a non-classical right boundary condition is given, where the accuracy provided by the method in interior nodes can be maintained at the boundary nodes.

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