International Journal of Innovative Research in Engineering & Management (IJIREM)
ISSN: 2350-0557, Volume-7, Issue-1, January 2020
https://doi.org/10.21276/ijirem.2020.7.1.1
www.ijirem.org

An Efficient and Accurate Hybrid Method for 1-D Wave Problem with Non-Classical Boundary Conditions

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ABSTRACT
A new numerical method is proposed for the solution of 1-D initial-boundary value wave problem with non-classical boundary conditions. It combines the finite difference method with that of characteristics. The second method operates near the boundary of integration domain and it implements accurately the non-classical boundary conditions. Meantime, the difference method operates efficiently and accurately in the interior of the domain. The key element is the achieving of a satisfying balance between accuracy and efficiency. The method is implemented and tested in Matlab.

Keywords-
Initial-boundary value problem, Hyperbolic differential equation, Damping conditions, Difference methods, Characteristics, Conventional order, Accuracy-efficiency balance.

1. INTRODUCTION
Overhead transmission lines, suspension bridges and pedestrian footbridges, suspended cables, loaded helical springs, and many other flexible structures are subject of oscillations, vibrations and other undesired movements due to different causes. To control the behavior of oscillations, and so to suppress their undesired effects, various types of dampers are applied in practice, (see [7]). The damping dashpot systems are usually installed in the two both ends of the flexible structure. This class of engineering problems in many cases can be modeled by wave equations with known initial conditions and specific boundary conditions, these last depended on the type of dampers applied and often referred as damping or viscose conditions, (see [2, 3, 4]). We will consider here the non homogenous wave equation

$$u_{tt} - u_{xx} = F(x, t) \tag{1}$$

with initial conditions

$$u(x,0)=f(x),\ u_t(x,0) = g(x), \quad 0 \le x \le l, \tag{2}$$

and non-classical damping conditions

$$u'_t(0, t) = \alpha_1 u'_x(0, t), \quad u'_t(l, t) = -\alpha_2 u'_x(l, t), \quad 0 \le t \le T \tag{3}$$

More complicated damping conditions are described in literature, as in [2] and [7].We will suppose here for simplicity that damping parameters α_1 and α_2 are equals to $\alpha > 0$.

When a damping dashpot system is to be installed in practice, the finding of optimal value for damping parameters α_1 and α_2 may require the numerical solution of problem (1-3) in a domain D=[0, l]x[0, T] of variables x and t and for different simulated initial conditions. The Energy Function, E(t) and the Total (negative) Energy of the system, TE, defined respectively as

$$E(t) = \int_0^l \left(u_t^2 + u_x^2\right)dx, \quad TE = \int_0^T E(t)dt,$$

maybe also required. Both E(t) and TE are strongly depended on damping parameters α_1 and α_2.

In this context, for each numerical method in solving problem (1-3), the performance of computational time processing is a real challenge. Efficiency and accuracy are important topics for each numerical computation, but often they are in an acute balance. The explicit second order Finite Difference Method, (hereafter FDM), is efficient in solving problem (1-2), but it fails to implement accurately the non-classical conditions (3). Meantime, the characteristic method can treat successfully these type of conditions, as in [1, 5], but its application in the whole domain D is very time consuming, compared to FDM.

An efficient and accurate numerical method is proposed in this paper to solve the problem (1-3). It combines the two above-mentioned methods applied in the specific squared grid determined by the method of characteristics. The basic idea and general algorithm will be presented in next section. In section 3 we implement the method in Matlab and a specific example of problem (1-3) will be solved both by FDM and new proposal. The global relative errors and conventional orders of accuracy will be estimated for each case. Efficiency of the new proposal, expressed in terms of computer processing time (Elapsed time), will be compared to that of FDM. Some extensions will be made in section 4.

2. AN HYBRID METHOD FOR THE SOLUTION OF PROBLEM (1-3)
Denote by D=[0 l]x[0 T] the physical space of variables x and t in (1-3). A square grid G_0 is obtained as it follows: Divide the interval $0 \le x \le l$ into m subintervals each of width h = l/m. Denote N = round(T/h) and $\bar{T}$ = hN. Redefine for convenience $T = \bar{T}$. The new interval $0 \le t \le T$ is divided into N subintervals each of width h = T / N. The grid G_0 where the solution of problem (1-3) is computed is presented in Figure 1.

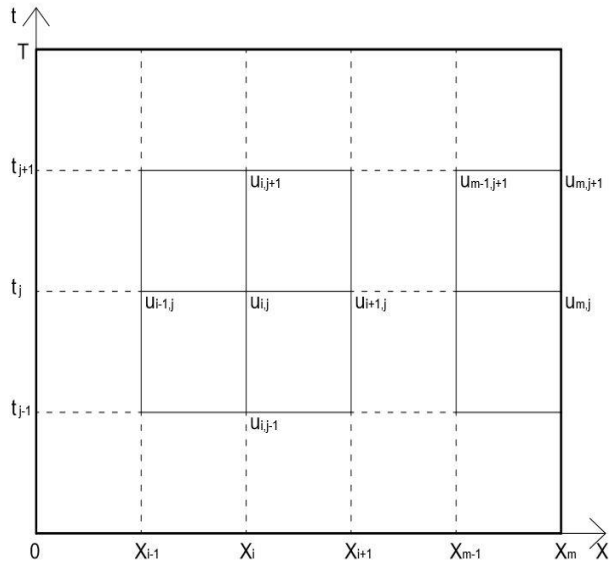

Figure 1: Difference method - Scheme of nodes where the solution of (1-3) is computed

The basic equations of the second order Finite Difference Method are simplified in this square grid.

$$u_{i,0}=f(x_i),\ i = 0,1,...,m; \tag{4}$$

$$u_{i,1}=f(x_{i-1})+ f(x_{i+1})+kg(x_i)+\frac{h^2}{2}F(x_i,0) \quad i = 1,...,m-1; \tag{5}$$

$$u_{i,j+1}=u_{i-1,j}+ u_{i+1,j}-u_{i,j-1}+h^2F(x_i,t_i); \tag{6}$$

j = 1, 2, ..., N, i = 1, 2, ... , m-1.

It looks difficult to find formulas of second order approximation for the implementation of boundary conditions (3). The equation

$$u'_t(l, t_{j+1}) = -\alpha u'_x(l, t_{j+1}), j=0,...N-1.$$

can be approximated using first order upward differences as

$$u(l,t_{j+1}) - u_{m,j}=-\alpha(u(l,t_{j+1}) - u_{m-1,j+1}).$$

So, $u(l,t_{j+1})$ can be approximated as

$$u_{m,j+1}=(u_{m,j} + \alpha u_{m-1,j+1})/(1+\alpha), \quad j = 0, 1, ..., N-1. \tag{7}$$

It can be verified that for the left boundary nodes in Figure 1, the above equation can be written as

$$u_{0,j+1}=(u_{0,j} + \alpha u_{1,j+1})/(1+\alpha), \quad j = 0, 1, ..., N-1. \tag{8}$$

Simplicity and efficiency of methods (4-8) is devaluated from the decreased accuracy of approximations (7) and (8). These equations give only O(h) approximations, while the local truncation error for equations (5) and (6) is $O(h^2)$. The method below will repair this drawback. The square grid G_0 is maintained as above, but the domain D is divided in three regions as in Figure 2. In the following we agree that the boundary between D_1 and D_2 belongs to D_1, and the boundary between D_2 and D_3 belongs to D_3. The characteristic method will be adopted for the solution of the problem (1-3) in the left D_1 and the right D_3 regions of Figure 2. Meantime, the FDM (4-6) will be applied in the central region D_2. Both two methods will cooperate properly in their boundaries.

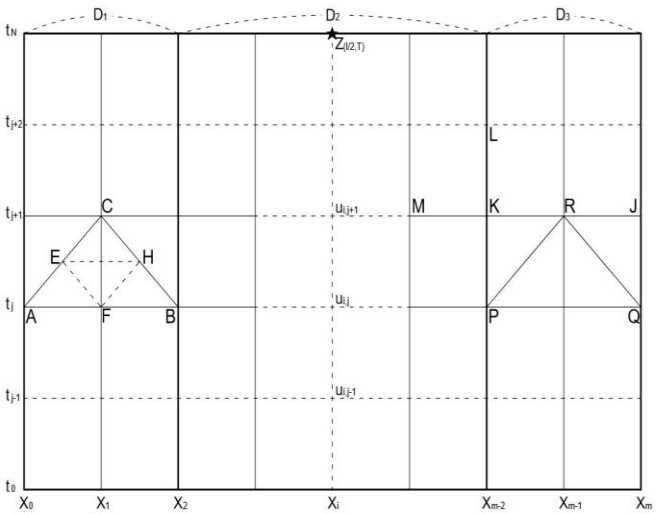

Figure 2: Hybrid method - Scheme of nodes and the regions where the solution of (1-3) is computed

It is assumed that $f'(x)$ exists so that $u_x(x,0) = f'(x)$, whenever $0 \le x \le l$. Consequently, $u(x,t)$, $u_x(x,t)$ and $u_t(x,t)$ will be known functions whenever $0 \le x \le l$ and $t = 0$.

We will explain the new proposal through the following algorithm:

Step (0): Compute $u_{i,0}= f(x_i),\ i = 0,1,...,m;$

Compute the derivatives of the solution u'_x and u'_t at the three first nodes $0 = x_0$, x_1, x_2 and at the three last nodes x_{m-2}, x_{m-1}, x_m, $= l$, using the initial data.

General step (j+1), j =1,2, ... N-1;

Suppose that up to level t_j, the solution is computed in the region D_2.

Suppose also that the solution u and its partial derivatives u'_x and u'_t are also computed up to level t_j in the regions D_1 and D_3.

Compute $u_{i,j+1}= u_{i-1,j}+ u_{i+1,j} - u_{i,j-1}+ h^2F(x_i,t_i).$ (9)

It can be verified that the (P,Q,R) process of the method of characteristics, applied for the problem (1-2), for the nodes P, Q and R of Figures 2, leads to the following system of equations:

$$u'_x(R)=\frac{1}{2}\left[u'_x(Q) + u'_x(P) + u'_t(Q) - u'_t(P)\right]+\frac{h}{4}\left[F(Q) - F(P)\right] \tag{10}$$

$$u'_t(R)=\frac{1}{2}\left[u'_x(Q) - u'_x(P) + u'_t(Q) + u'_t(P)\right]+\frac{h}{4}\left[F(Q) +2F(R) + F(P)\right] \tag{11}$$

$$u(R)=\frac{1}{2}\left[u(P) + u(Q)\right]+\frac{h}{4}\left[u'_x(P) - u'_x(Q) + u'_t(P) + 2u'_t(R) + u'_t(Q)\right] \tag{12}$$

Knowing the values of u and its partial derivatives at points R and Q we can receive the approximation

$$u(J) =\frac{1}{2}\left[(u(R) + u(Q))+h(u'_x(R) + u'_x(Q))\right].$$

It can be proved that the local error of this approximation is $O(h^2)$

Having the values of u at points K, R and J, the derivative $u'_x(J)$ can be approximated using a three-point backward formula

$$u'_x(J) = \frac{1}{2h}\left[3u(J) - 4u(R) + u(K)\right].$$

Compute the derivative $u'_t(J)$ as

$$u'_t(J) = -\alpha\, u'_x(J).$$

Compute

$$u(L) = u_{m-2,j+2} = u_{m-3,j+1} + u_{m-1,j+1} - u_{m-2,j} + h^2 F(x_{m-2}, t_{j+1}).$$

and then use the centered-difference quotient for the first partials

$$u'_x(K) = \frac{1}{2h}\left[u(R) - u(M)\right], \quad u'_t(K) = \frac{1}{2h}\left[u(L) - u(P)\right].$$

It can be seen that the general step for the points P, Q, R, J, K, L, and M of region D_3 can be extended symmetrically to the region D_1. So, at the end of step (j+1), the solution u is computed up to level t_{j+1}, in the region D_2. Also, the solution u and its partial derivatives u'_x and u'_t are computed up to level t_{j+1}, in the regions D_1 and D_3. It can be realized also how the general step can be modified as the first step, (case j=0).

3. MATLAB IMPLEMENTATION AND NUMERICAL RESULTS

The Proposed Hybrid Method, hereafter referred as PHM, described in section 2, is implemented in Matlab.

Consider the initial value non-homogeneous wave problem

$$u_{tt} - u_{xx} = x + t \qquad (13)$$

with initial conditions

$$f(x) \equiv u(x,0) = x, \quad u_t(x,0) \equiv g(x) = (-2x+1)\alpha, \quad 0 \le x \le 1 \qquad (14)$$

and boundary condition

$$u'_t(0, t) = \alpha u'_x(0, t), \quad u'_t(1, t) = -\alpha u'_x(1, t), \quad 0 \le t \le T. \qquad (15)$$

It is very difficult, may be impossible, to obtain analytical solution for the problem (1-3), see e.x. [6].

Initial-Boundary value problem (13 - 15) is first solved by PHM for the values T=10 and α = 0.7. Approximations of the solution are computed at the point Z(0.5, 10), for some typical values of stepsize h = 1/m, selected for good performance. In one version the values of m are selected progressively, $m = 2^n$, n = 4, 5, ... ,13. In the other version m is distributed uniformly in the interval [500 5000]: m = 500n, n = 1, 2, ... ,10. The results of computations are presented in Table 1. Different display format are applied (for variation), for the two versions above. We can see in Table 1 the convergence of the results as m increases. The value U (bold in Table 1), received for m = 8192, can be considered as the best approximation for the theoretic value u(0.5, 10). But the computation with m = 8192 would be very time consuming for several experiments done below. We can see from Table 1, that the results in long format of the six last rows have six idem significant digits. Meantime the result for m = 1000 (bold in Table 1), has 5 idem significant digits with the best result.

Table 1: Results of computations for PHM solution at point Z(0.5, 10), for T=10 and α=0.7

m=2^n	u(0.5, 10)	m	u(0.5, 10)
16	1.9686e+01	500	19.744916865142972
32	1.9721e+01	**1000**	**19.745567877202518**
64	1.9735e+01	1500	19.745782535786446
128	1.9741e+01	2000	19.745889425215140
256	1.9744e+01	2500	19.745953418169201
512	1.9745e+01	3000	19.745996021368306
1024	1.9746e+01	3500	19.746026423660165
2048	1.9746e+01	4000	19.746049209556425
4096	1.9746e+01	4500	19.746066922719006
8192	**1.9746e+01**	5000	19.746081087268507

Really, the value m = 1000 is at least 10 times more than usual values of m applied in integration practice. In addition to this, we aim to achieve a satisfying balance between accuracy and efficiency. In this context, in lack of the analytic solution for the problem, we will accept and use below the value U received by m = 1000, as exact value of the solution. To compare different results, it make sense to compute and use relative errors instead of absolute ones.

The problem (13-15) is solved by FDM (4-8) and by PHM, for T = 10 and m = 10n, n = 1, 2, ... , 10. The relative global errors (u(m) - U) / U, at point Z(0.5, 10), are computed in each case, for both two methods FDM and PHM. These errors are presented in the Table 2 below. From this table it can be seen that the global error of PHM are 25-60 times smaller than those of FDM.

Table 2: Relative global errors for FDM and PHM solutions at point Z (0.5, 10), for α=0.7 and T = 10.

m	MDF	PHM
10	7.7561e-02	5.8353e-03
20	3.8416e-02	2.2411e-03
30	2.5523e-02	1.3365e-03
40	1.9105e-02	9.3833e-04
50	1.5264e-02	7.1770e-04
60	1.2707e-02	5.7786e-04
70	1.0882e-02	4.8161e-04
80	9.5149e-03	4.1140e-04
90	8.4520e-03	3.5796e-04
100	7.6020e-03	3.1596e-04
r	1.0081e+00	1.2540e+00
c	7.8841e-01	9.8834e-02

Considering the general expression of global error:

$$g = ch^r \qquad (16)$$

and using the error data of Table 2, we can compute the value of c and r per each of the method used. In the two last row of Table 2 are displayed the computed values of r and c for FDM and PHM. We may not be surprised for the poor accuracy of FDM. Its constant of error c is about 8 times more than that of PHM, and its conventional order of accuracy, r = 1.01, is lower compared to that of PHM. The poor results of FDM were expected considering the equations (7) and (8) of this method, which give only O(h) approximations. A detailed discussion on the accuracy orders of PHM and FDM, and their constants of errors, will be done below.

Reasoning physically, it can be assumed that the effects of boundary conditions (3) on the solution u at point Z, denoted by symbol ★ in Figure 2, are proportional to the ratio T/l, and for given l, they are proportional to T. So, the errors at point Z, due to approximations (7-8) of FDM, or several ones of PHM, will be depended on T. The problem (13-15) is solved by FDM and PHM for a range values of T, and in each case the values r and c are computed. The corresponding results are presented in Table 3 below.

Table 3: Results for r and c at (16), for FDM and PHM solutions at point Z (0.5, T), for different T.

	FDM		PHM	
T	**r**	**c**	**r**	**c**
0.4	2.004	0.12	2.004	0.12
1	2.111	0.15	1.859	0.42
2	1.057	0.15	1.760	0.41
5	1.016	0.59	1.409	0.17
10	1.008	0.79	1.254	0.10
20	1.005	0.90	1.158	0.07
50	1.003	0.96	1.090	0.05
100	1.002	0.98	1.070	0.05

We could read through table 3:

For values of T < 0.5, the triangle region of determinacy of point Z does not intersect the boundaries, so the solution is not influenced by the boundary conditions. FDM and PHM operate in this case like a FDM for an initial-value problem. That's why r and c, for T=0.4, are the same for both two methods, and their order of accuracy are approximately to 2, which is the theoretic local order of FDM. As T increases the accuracy order of FDM converges rapidly to one (theoretic global order of FDM), and its constant of error c increases. On the contrary, the constant c of PHM decreases as T increases (for T≥1), meantime r converges to one, but in a slower rate compared to that of FDM. In typical application of wave equation, the value of T is moderate, so PHM if superior on FDM.

In another numerical experiment problem (13-15) is solved for T = 10 by FDM and PHM, and for a range values of h = l/m. Elapsed times and the ratio R of the corresponding elapsed times, are computed in each case. The results of this numerical experiment are presented in Table 4 below.

Table 4: Elapsed time in seconds and the ratio R

m	**10**	**100**	**500**	**1000**	**2000**
FDM	0.0056	0.0918	2.2592	9.0767	36.7187
PHM	0.0563	0.1144	2.3704	9.3363	37.4639
R	10.13	1.25	1.05	1.03	1.02

We could read through Table 4:

For m = 10, FDM is about 10 times faster than PHM, but for t = 2000, it is only 1.02 times faster. This was expected: As m increases the central domain D_2 "goes" to D (domains D_1 and D_3 diminish). Clearly the ratio R goes to one as m goes to infinity. Reading Tables 2 and 4, for the typical value m = 100, we could say: By using PHM instead of FDM for solving problem (13-15), we may increase with 25 percent the amount of calculation, but the error of computation could be decreased up to 24.06 times.

In all experiments done in solving problem (1-3) for different functions F(x, t), f(x), g(x) and different α, the corresponding tables look similar to Tables 1,2,3, and 4 above, and the analyses done above stand.

4. EXTENSIONS

It could be seen that processes (P, Q, R), (9-11) of PHM operate in regions D_1 and D_3 with stepsize 2h, meantime PHM operates in region D_2 like FDM with stepsize h. Certainly this reduces the accuracy of PHM. Significant improvement in accuracy can be achieved if the method of characteristics developed in [1], would be adopted in section 2, instead of classic method of characteristics. Referring to the Figure 2 of this paper, the process (A, B, C) and triple processes (A, F, E), (F, B, H) and (E, H, C), are used for the approximation of the solution at point A. According to [1], the order of accuracy would be improved by one unit. This would surely increase the general accuracy of PHM. The new process above seems to be four times more expensive than simple one, but considering that regions D_1 and D_3 are extended in one dimension, the amount of calculation and physical memory storage would be increased insignificantly.

PHM can be extended for the case of semi linear hyperbolic equation. By proper variable transformations this equation can be transformed into canonical form (2), but with $F(x, t, u, u_x, u_t)$ instead of F(x, t). In this case, after substituting u_x and u_t with the centered-difference quotient, the equation (9) would become an implicit one, and (11-12) would become an implicit system of equations.

5. CONCLUSIONS

In this paper an hybrid numerical method has been proposed and developed for the solution of initial boundary value problem (1-D wave equation), with non-classic boundaries conditions. The domain D of integration is divided into three regions, where the central region occupies the largest part of D. The proposed method operates as classic FDM in the central region, and as method of characteristics in the two lateral regions. The method was implemented in Matlab and some numerical experiments has been done using specific examples. It has been observed that the new proposal increases considerably the accuracy with an insignificant increasing of the amount of calculation and physical memory storage. The problem can be solved by economic effort, and the extra elapsed time compared to classic difference method, is fully justified by the accuracy provided.

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