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A Non-Isomorphic Singular Group Divisible Design

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ABSTRACT- This paper proposes a new technique for finding a series of Group Divisible Designs (GDDs), starting from an existent Balanced Incomplete Block Design (BIBD). Concerning the parameters, the series is the same as that proposed by Bose and Connor [3]. However, the proposed GDD provides a non-isomorphic design to the S 28, Clathworthy [2] with $v = 12, b = 10, r = 5, k = 6, \lambda_1 = 5, \lambda_2 = 2; m = 6, n = 2$.

KEYWORDS- Non-Isomorphic Block Designs, Singular Group Divisible Design, Balanced Incomplete Block Design.

I. INTRODUCTION

Bose and Connor [3] have constructed a series of Singular Group Divisible Design (SGDD) with parameters $v = nv^*$, $b = b^*$, $r = r^*$, $k = nk^*$, $\lambda_1 = r^*$, $\lambda_2 = \lambda^*$; $m = v^*$, n, starting from BIBD (v^*, b^*, r^*, k^*, λ^*) by replacing each treatment with a group of n treatments. Also, many relations among the parameters of the design have been proposed for the existence of GDDs. Dey and Nigam [6] have suggested a construction-method of GDD($v = v^*/s$, $b = b^*/s$, $r = r^*$, $k = k^*$, λ_1=0, $\lambda_2 = s\lambda_2^*$; $m = m^*$, $n = t$), starting from another GDD($v^* = m^*n^*$, b^*, r^*, k^*, $\lambda_1^* = 0$, $\lambda_2^* = m^*$, $n^* = st$, $s \geq 2, t \geq 2$). In the literature of Bose, Shrikhande and Bhattacharya [4] many series of GD designs are presented by using combinatorial methods and orthogonal arrays. Kageyama and Tsuji [5] have contributed that (i) a GDD is Semi-regular iff k/m is an integer and block contains exactly k/m treatment(s) from each group of the association scheme and (ii) a GDD is Singular iff k/n is integer and every block contains exactly k/n groups of the association scheme. Kumar[9] has proposed a method of construction of GD designs from a given BIB design, using Kronecker products which were introduced by Vartak [12] where the constructed PBIB designs, even though they are less efficient than the parent BIB design, can exist for different combinations of "r" and "k". Meitei [8] has constructed a series of Semi-Regular GD (SRGD) design from a parent SRGD design by increasing block size by 1 and treatment number by n. Recently, enforcing and incorporating neighbour effects in Group Divisible design, Manjunatha et al. [7] introduced Generalized Neighbour Designs based on Group Divisible Association Scheme (GDAS), in circular block. Thus, the importance of GD design is being extended in the experiment where neighbour effect of treatments inherits. The recent contribution of Saurabh and Sinha [10] [11].

Definition- Two block designs D_1 and D_2 with the same parameters v, b, r, k are said to be isomorphic if there exists a bijective mapping f: $D_1 \to D_2$ (or f: $D_2 \to D_1$). Otherwise, they are said to be non-isomorphic.

By the definition of isomorphic designs, two block designs with same parameters, are isomorphic if relabelling the treatment names of one of the two block designs, another design can be obtained. Further, when the repetition of cardinalities c_{1j}(j=1,2, …,b(b-1)) of intersections of all 2 blocks of D_1 are not same to that of cardinalities c_{2j} of intersections of all 2 blocks of D_2, there does not exist any bijective mapping f such that f: $D_1 \to D_2$ (or, f: $D_2 \to D_1$).

II. SINGULAR GROUP DIVISIBLE DESIGNS

Using the incidence matrix of a BIBD, a series of GDDs, which gives a non-isomorphic design to S 28, Clathworthy [2], can be constructed as given below:

Theorem 2.1 The existence of a BIBD (v^*, b^*, r^*, k^*, λ^*) implies that of SGD ($v = sv^*$, $b = b^*$, $r = r^*$, $k = sk^*$, $\lambda_1 = r^*$, $\lambda_2 = \lambda^*$; $m = v^*$, $n = s$)

Proof: Let $N^* = (n_{ij}^*)$ be the incidence matrix of BIBD (v^*, b^*, r^*, k^*, λ^*); $i = 1, 2, \dots, v^*$; $j = 1, 2, \dots, b^*$. Then, $\sum_{i=1}^{v^*} n_{ij}^* = k^*$, $\sum_{j=1}^{b^*} n_{ij}^* = \sum_{j=1}^{b^*} (n_{ij}^*)^2 = r^*$ and $\sum_{j=1}^{b^*} n_{ij}^* n_{i'j}^* = \lambda^*$ for $i \neq i'$ (2.1)

Consider N as the incidence matrix of the desired design and as given by

$N = N^* \otimes J_s$, where $\otimes$ and J_s denote the Kronecker product and $s \times 1$-matrix containing "1" only

$= (n_{ij}^*) \otimes J_s = (n_{pv^*+i,j})$, say; $p = 0, 1, \dots, s-1$. For fixed i, j; $n_{pv^*+i,j} = n_{ij}^*$ $\forall p$ … (2.2)

Since the order of the incidence matrix N is $sv^* \times b^*$, then, $v = sv^*$, and $b = b^*$.

Now, $\sum_{j=1}^{b^*} n_{pv^*+i,j} = \sum_{j=1}^{b^*} n_{ij}^* = r^*$ $\forall p$ by (2.1) and

$\sum_{p=0,i=1}^{s-1,v^*} n_{pv^*+i,j} = \sum_{i=1}^{v^*} n_{i,j} + \sum_{i=1}^{v^*} n_{v^*+i,j} + \dots + \sum_{i=1}^{v^*} n_{(s-1)v^*+i,j}$

$= s\sum_{i=1}^{v^*} n_{ij}^* = sk^*$ by (2.2) and (2.1).

Hence, $r = r^*$ and $= sk^*$.

Further, $\sum_{j=1}^{b^*} n_{pv^*+i,j} n_{qv^*+i,j} = \sum_{j=1}^{b^*} (n_{ij}^*)^2$ for $p \neq q \in \{0, 1, \dots, s-1\}$ by (2.2)

$= r^*$ by (2.1) … (2.3)

and $\sum_{j=1}^{b^*} n_{pv^*+i,j} n_{qv^*+i',j} = \sum_{j=1}^{b^*} n^*_{ij} n^*_{i'j}$ for $i \neq i'$; $p, q \in \{0, 1, \ldots, s-1\}$
$= \lambda^*$ by (2.1). .... (2.4)
Treating the treatments, $i, i+v^*, \ldots\ldots, i+(s-1)v^*$ as the elements in the $i-th$ group of a Group Divisible Association scheme, as $i = 1, 2, \ldots, v^*$, clearly, $\lambda_1 = r^*$, $\lambda_2 = \lambda^*$, $m = v^*$, $n = s$, by (2.3) and (2.4).
Using the BIBD given in the Sr. No. 1, page 362 of Takeuchi[1], an illustration of the Theorem 2.1 follows.

Example 2.1 The blocks of the BIBD($v^* = 6$, $b^* = 10$, $r^* = 5$, $k^* = 3$, $\lambda^* = 2$), (∞, 0, 1), (∞, 1, 2), (∞, 2, 3), (∞, 3, 4), (∞, 4, 0), (0, 1, 3), (1, 2, 4), (2, 3, 0), (3, 4, 1), (4, 0, 2) with the incidence matrix, treating "∞" as the 6th treatment,

$$N^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Taking s=2, the incidence matrix of the SGD ($v = 12, b = 10, r = 5, k = 6, \lambda_1 = 5, \lambda_2 = 2$; $m = 6, n = 2$) based on the Group Divisible Association scheme, $G_i = \{i, 6+i\}$; $i = 1, 2, \ldots, 6$, is given by $N = N^* \otimes J_2$, having the blocks, B_1=(1, 2, 6, 7, 8, 12), B_2=(2, 3, 6, 8, 9, 12), B_3=(3, 4, 6, 9, 10, 12), B_4=(4, 5, 6, 10, 11, 12), B_5=(1, 5, 6, 7, 11, 12), B_6=(1, 2, 4, 7, 8, 10), B_7=(2, 3, 5, 8, 9, 11), B_8=(1, 3, 4, 7, 9, 10), B_9=(2,4, 5, 8, 10, 11), B_{10}=(1, 3, 5, 7, 9, 11).

Remark 2.1 The above SGDD is non-isomorphic to the design, S-28 [2].
Verification: The blocks of the design, S 28 are B_1=(1, 7, 2, 8, 3, 9), B_2=(2, 8, 3, 9, 4, 10), B_3=(3, 9, 4, 10, 5, 11), B_4=(4, 10, 5, 11, 1, 7), B_5=(5, 11, 1, 7, 2, 8), B_6=(5, 11, 2, 8, 6, 12), B_7=(1, 7, 3, 9, 6, 12), B_8=(2, 8, 4, 10, 6, 12), B_9=(3, 9, 5, 11, 6, 12), B_{10}=(4,10, 1, 7, 6, 12).

Among the cardinalities of all $\binom{10}{2}$ i.e. 45 possible intersections of 2 blocks of the design S-28, the cardinality 4 replicates 14 times and the cardinality 2, 31 times. On the other side, among the cardinalities of all $\binom{10}{2}$ i.e. 45 possible intersections of 2 blocks of the design in the Example 2.1, the cardinality 4 replicates 15 times and the cardinality 2, 30 times. So, either of the 2 designs cannot be mapped from another by any bijective mapping.

III. CONCLUSION

The above theorem and remark with verification is based on the Clathworthy [2], S-28 on SGDD is isomorphic.

CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest.

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