Unique Mathematical Technique Applied To the Bifurcation Phenomena Using Catastrophe Theory

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ABSTRACT
This manuscript presents a new technique to derive an accurate describe of sporadically perturbed primary bifurcations in two non-linearly coupled oscillators in both the non-resonant and resonant cases. Statistical methodologies traditionally used by behavioral scientists assume that variables are continuous and linearly related. (Thom.C., 1975) has shown mathematically that catastrophe theory is an appropriate methodology to examine discontinuous non-linear relationships. While Thorn developed the theory to deal primarily with biological problems, (Zeeman, 1975) has explained how this approach can be effective with behavioral science phenomena. Catastrophe theory allows examination of discontinuous effects occurring over time.The bifurcations in the subharmonic resonant case are Z_2-symmetry ones and they have codimension 3. On the other hand, the bifurcations in the main resonant case are shown to possess no symmetry properties and their codimension increases to 5. These bifurcation problems are analyzed in detail and the quasiglobal bifurcation classifications as well as bifurcation diagrams in the system are presented. It is shown that one can control vibration in non-linear systems with an appropriate choice of system parameters as suggested by some regions in the hyper surface where the amplitude of bifurcating solution is always zero.

Keywords: Bifurcation Phenomena, Statistica, methodologies, Non-Linearly Coupled Oscillator, Primary Bifurcations, Secondary Bifurcations, Catastrophe Theory.

1. INTRODUCTİON
Considerable interest has been paid to the bifurcation phenomena of non-linear oscillating systems in engineering applications over the past five years [11. These phenomena include jump responses, hysteresis responses, periodic motions, quasiperiodic motions, sub-harmonic and super harmonic oscillations and chaotic motions. Many new mathematical ideas and theories, for instance the theory of Hopf bifurcation [2], the method of averaging [3-61, the centre manifold theorem [7], the theory of normalforms [7], and alternativeapproaches [6] have played an important role in the study of bifurcation problems. With the development of bifurcation studies, typical bifurcations or simple bifurcationssuch as the Hopf bifurcation, the saddle-node bifurcation and the pitchfork bifurcationhave become quite familiar. However, more complicated bifurcations or highercodimension bifurcations have not yet received much attention.The influence of a periodic perturbation on a bifurcating system has been studied by Rosenblatt and Cohen [S, 91 using the method of multiple scales. Similar studies have been made by Kath [l0], where periodic perturbations appear as external disturbances. Bajaj [ll] has studied some of these problems using the alternative method. Recently, Namachchivaya and Ariaratnam [12] have studied periodically perturbed Hopf bifurcation using the averaging method, normal forms and the centre manifold theorem. However, the investigations mentioned above have been focused only on the bifurcation problems originating from non-linear dynamic systems with one degree of freedom. The important issues of

i- higher codimension bifurcations
ii- the global properties of the bifurcations in parameter space were not considered.

These issues can be addressed by using the theory of singularity and group methods in bifurcation research [13]. In this paper a non-linear dynamic system with two degrees of freedom is considered so as to provide an initial framework for similar analysis of systems with many degreesof freedom. The system consists of two non-linearly coupled oscillators with small parametric excitation and external force. Only primary bifurcations are considered. The associated secondary bifurcation problems will be dealt with in a companion paper [141. Equations of the type studied in this paper arise from the investigation of finite amplitude oscillations of hinged-clamped columns [15] subjected to an axial harmonic load or pipesconveying a moving fluid. They can also be found in biological studies in which the equations describe "states", for example the concentration of chemical species of two neigh boring cells or groups of cells, each of which is able to oscillate by itself [16] and also if solutes are permitted to flow between these oscillators by diffusion and they areaffected periodically by external factors such as temperature and humidity, etc. In what follows the averaging method [3-5] is applied to study periodically perturbed primary bifurcations of the system and to obtain the bifurcation equations. A description of the dynamic model is made. Periodically perturbed primary bifurcations in the non-resonant cases are examined. The influence of parametric force and external force on the primary bifurcations is shown. Primary bifurcations in sub harmonic resonance associated with parametric excitation are studied. The symmetry bifurcation problem is analysed in detail. Primary bifurcations in main resonance associated with an externally applied force are investigated .This leads to a different kind of bifurcation problem from that of section 4. Some quasiglobal bifurcation pictures are presented in every section. It should be noted that these quasiglobal bifurcation pictures could not be obtained by previous techniques included in the aforementioned references.

2. CATASTROPHE THEORY

Catastrophe theory, in mathematics, a set of methods used to study and classify the ways in which a system can undergo sudden large changes in behaviour as one or more of the variables that control it are changed continuously. Catastrophe theory is generally considered a branch of geometry because the variables and resultant behaviours are usefully depicted as curves or surfaces, and the formal development of the theory is credited chiefly to the French topologist René Thom.A simple example of the behaviour studied by catastrophe theory is the change in shape of an arched bridge as the load on it is gradually increased. The bridge deforms in a relatively uniform manner until the load reaches a critical value, at which point the shape of the bridge changes suddenly—it collapses. While the term catastrophe suggests just such a dramatic event, many of the discontinuous changes of state so labelled are not. The reflection or refraction of light by or through moving water is fruitfully studied by the methods of catastrophe theory, as are numerous other optical phenomena. More speculatively, the ideas of catastrophe theory have been applied by social scientists to a variety of situations, such as the sudden eruption of mob violence.Catastrophe theory analyses degenerate critical points of the potential function — points where not just the first derivative, but one or more higher derivatives of the potential function are also zero. These are called the germs of the catastrophe geometries. The degeneracy of these critical points can be unfolded by expanding the potential function as a Taylor series in small perturbations of the parameters.When the degenerate points are not merely accidental, but are structurally stable, the degenerate points exist as organising centres for particular geometric structures of lower degeneracy, with critical features in the parameter space around them. If the potential function depends on two or fewer active variables, and four or fewer active parameters, then there are only seven generic structures for these bifurcation geometries, with corresponding standard forms into which the Taylor series around the catastrophe germs can be transformed by diffeomorphism (a smooth transformation whose inverse is also smooth).[citation needed] These seven fundamental types are now presented, with the names that Thom gave them

2.1 Catastrophe Theory and Bifurcation Analysis

Statistical methodologies traditionally used by behavioral scientists assume that variables are continuous and linearly related. (Thom.C., 1975) has shown mathematically that catastrophe theory is an appropriate methodology to examine discontinuous non-linear relationships. While Thorn developed the theory to deal primarily with biological problems, (Zeeman, 1975) has explained how this approach can be effective with behavioral science phenomena. Catastrophe theory allows examination of discontinuous effects occurring over time.Catastrophe theory is not concerned with the examination of tragedies as would be surmised from the popular use of the word, but pertains to sudden and abrupt discontinuous change. This may refer to values, opinions or behavior which does not occur via a smooth transition. (Jiobu and Lundgren, 1978) explain that there are theoretically seven possible models from the "fold" (one dependent and one independent variable) to the parabola (two dependent and four independent variables). The applicable model for this study is the cusp model which has one dependent and two independent variables. The other models (except for the "fold") are more complicated, extremely difficult to conceptualize, and even more difficult to operationalize. The cusp model is visualized in Figure (1). The model is usually conceptualized as being four dimensional even though only three dimensions can be drawn. The values for the two independent variables *(X, Z)* are presented on the "control surface."The dependent variable (*Y*) is the height of the figure and is displayed on the "behavior surface." This surface has different heights which indicate the extent of the dependent variable at each juncture of the two independent variables. The fourth dimension concerns the variables over time. The cusp model is characterized by four attributes (Flay, 1978). First, the behavior can be considered bimodal dependent upon different values of the control factors (independent variables). Abrupt changes are observed between one mode of the dependent variable and another. This change is characterized by a delay rule where the behavioral variable tends to be continuous and in one behavioral state as long as possible (Sussman and Zahler, 1978). The cusp-catastrophe model therefore describes a phenomenon where two continuous independent variables impact upon a dichotomous dependent variable. The dependent variable will remain in the same bimodal state until an abrupt change or catastrophe occurs (a result of the "delay rule") when the continuous independent variables reach some threshold point The "A" time path on the behavior surface in Figure 3.1 illustrates abrupt movement from one plane or mode of the dependent variable to another. The threshold is reached as the independent variables or factors cross the shaded area in Figure 3.1 (depicted by "a" on the behavior surface). The trace of this behavioral path is projected on the control surface (time path "A" on the control surface). The second attribute is the shaded area or inaccessible region on the behavior surface. This occurs since theoretically people cannot be actively involved in two contradictory behaviors simultaneously (i.e., people either stay or leave an organization). This area is represented by the bifurcation plane on the control surface. Movement across the bifurcation plane by the two independent control surface factors would therefore imply an abrupt change in behavior on the behavior surface. Third, there is hysteresis in the behavior surface implying that abrupt changes from one mode of behavior to another can occur at different values of the control surface depending on the direction of the time path. Any combination of control factors on the bifurcation plane represents a threshold beyond which intent to leave or termination behavior will occur. Not all time paths will necessarily cross the bifurcation plane. This possibility is illustrated in Figure (1) by the projections of the "B" and "C" time path on the control surface. If both "B" and "C" do not change, the behavioral state on the behavior surface will not change abruptly. The fourth attribute of divergent behavior concerns comparative changes in behavior. As one approaches the shaded area, even small changes in the control factors may result in catastrophic or abrupt changes in behavior. At other locations on the behavior surface large changes in perceptions may not influence behavior.

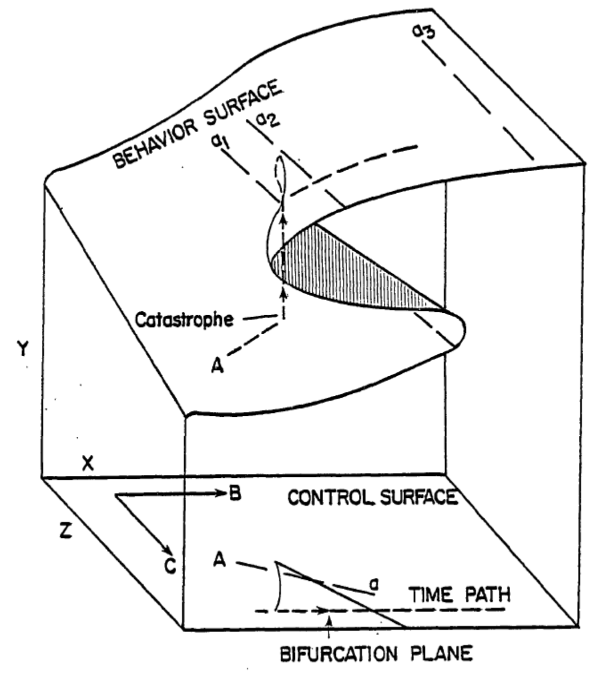

Figure 1: Model is usually conceptualized as being four dimensional even though only three dimensions can be drawn.

Catastrophe theory can be briefly described as follows. Consider a system whose. The behaviour is usually smooth but which exhibits some discontinuities. Suppose the system has a smooth potential function to describe the system dynamics and has "n" state variables and "m" control parameters. Given such a system, catastrophe theory tells us the following: The number of qualitatively different 'configurations of discontinuities that can occur depends not on the number of state variables but on the number of control parameters. Specifically, if the number of the control parameters is not greater than four, there are seven basic or elementary catastrophes, and in none of these are more than

Two state variables involved [10].

Consider a continuous potential function *V (Y, C)* which represents the system behavior, where *Y* are the state variables and *C* are the controlparameters. The potential function *V(Y,C)* can be mapped interms of its control variables *C* to define the continuous region. Let the potential function be represented by:

$$V(Y,C): M \otimes R \tag{1}$$

Where *M, C* are manifolds in the state space R^n and the control space R^r respectively.

Now we define the catastrophe manifold M as the equilibrium surface that represents all critical points of *V(Y,C)*. It is the subset $R^n \times R^r$ defined by:

$$\nabla_Y V_{c(Y)} = 0 \tag{2}$$

Where $V_{c(\mathrm{Y})} = V(Y,C)$ and $\nabla \mathrm{y}$ is the partial derivative with respect to *Y*. Equation is the set of all critical points of the function $V(Y,C)$.Next we find the singularity set, S, which is the subset of *M* that focus on all degenerate critical points of V. It is defined by:

$$\nabla_Y V_{c(Y)} = 0$$

And

$$\nabla_Y^2 V_{c(Y)} = 0 \tag{3}$$

The singularity set, S, is then projected down onto the control space R^r by eliminating the state variables x using, to obtain the bifurcation set, B. The bifurcation set provides a projection of the stability region of the function $V(Y,C)$, i.e. it contains all non-degenerate critical points of the function V bounded by the degenerate critical point at which the system exhibits sudden changes when it is subject to small changes.The seven elementary catastrophes of $r < 4$ are listed in [6]. The geometric analyses of the catastrophes that are used in this thesis are presented in detail in Appendix (A). A simplified analysis of the seven elementary catastrophes is given in [7].

Table 1: Seven Elementary Catastrophes

Catastrophe	Control Space Dimension	State Space Dimension	Function	Catastrophe Manifold
Fold	1	1	$\frac{1}{3}x^3 - ax$	$x^2 - a$
Cusp	2	1	$x^4 - ax - \frac{1}{2}bx$	$x^3 - a - bx$
Swallowtail	3	1	$\frac{1}{5}x^5 - ax - \frac{1}{2}bx^2 - \frac{1}{3}cx^3$	$x^4 - a - bx - cx^2$
Butterfly	4	1	$\frac{1}{6}x^6 - ax - \frac{1}{2}bx^2 - \frac{1}{3}cx^3 - dx^4$	$x^5 - a - bx - cx^2 - dx^3$
Hyperbolic	3	2	$x^3 + y^3 + ax + by + cxy$	$3x^2 + a + cy$ $+3y^2 + b + cx$
Elliptic	3	2	$x^3 - xy^2 + ax + by + cx^2 + cy^2$	$3x^2 - y^2 + a + 2cx$ $-2xy + b + 2cy$
parabolic	4	2	$x^2y + y^4 + ax + by + cx^2 + dy^2$	$2xy + a + 2cx$ $+x^2 + 4y^3 + b + 2dy$

2.2 Application of Cusp Catastrophe Theory and Mathematical Problem

The cusp geometry is very common, when one explores what happens to a fold bifurcation if a second parameter, b, is added to the control space. Varying the parameters, one finds that there is now a curve (blue) of points in (a,b) space where stability is lost, where the stable solution will suddenly jump to an alternate outcome.But in a cusp geometry the bifurcation curve loops back on itself, giving a second branch where this alternate solution itself loses stability, and will make a jump back to the original solution set. By repeatedly increasing b and then decreasing it, one can therefore observe hysteresis loops, as the system alternately follows one solution, jumps to the other, follows the other back, then jumps back to the first.However, this is only possible in the region of parameter space a < 0. As a is increased, the hysteresis loops become smaller and smaller, until above a = 0 they disappear altogether (the cusp catastrophe), and there is only one stable solution.One can also consider what happens if one holds b constant and varies a. In the symmetrical case b = 0, one observes a pitchfork bifurcation as a is reduced, with one stable solution suddenly splitting into two stable solutions and one unstable solution as the physical system passes to a < 0 through the cusp point (0,0) (an example of spontaneous symmetry breaking). Away from the cusp point, there is no sudden change in a physical solution being followed: when passing through the curve of fold bifurcations, all that happens is an alternate second solution becomes available.A famous suggestion is that the cusp catastrophe can be used to model the behaviour of a stressed dog, which may respond by becoming cowed or becoming angry.[1] The suggestion is that at moderate stress (a > 0), the dog will exhibit a smooth transition of response from cowed to angry, depending on how it is provoked. But higher stress levels correspond to moving to the region (a < 0). Then, if the dog starts cowed, it will remain cowed as it is irritated more and more, until it reaches the 'fold' point, when it will suddenly, discontinuously snap through to angry mode. Once in 'angry' mode, it will remain angry, even if the direct irritation parameter is considerably reduced.A simple mechanical system, the "Zeeman Catastrophe Machine", nicely illustrates a cusp catastrophe. In this device, smooth variations in the position of the end of a spring can cause sudden changes in the rotational position of an attached wheel.[2]Catastrophic failure of a complex system with parallel redundancy can be evaluated based on relationship between local and external stresses. The model of the structural fracture mechanics is similar to the cusp catastrophe behavior. The model predicts reserve ability of a complex system.Other applications include the outer sphere electron transfer frequently encountered in chemical and biological systems[3] and modelling Real Estate Prices.[4]

Fold bifurcations and the cusp geometry are by far the most important practical consequences of catastrophe theory. They are patterns which reoccur again and again in physics, engineering and mathematical modelling. They produce the strong gravitational lensing events and provide astronomers with one of the methods used for detecting black holes and the dark matter of the universe, via the phenomenon of gravitational lensing producing multiple images of distant quasars.In this example we have two control variables asb and one state variable x . In modern mathematics we don't recognize the existence of variables, though, so we have to introduce the control space. The potential function is:

$$V(X) = X^4 + uX^2 + vX \quad (4)$$

So the equilibrium surface is a three-dimensional space in x, u and v given by:

$$4X^3 + 2uX + v = 0 \quad (5)$$

First problem is that this expression was obtained by truncating a Taylor series to 4th order. It is conceivable that taking higher terms might destroy the shape of the equilibrium surface M . Certainly if we work only to 3rd order we get quite a different picture. For then we have a potentialand the singularity set is the subset of the equilibrium surface such that the derivative of (4) is also equal to zero. It is given by:

$$12x^2 + 2u = 0 \quad (6)$$

We find the bifurcation set by eliminating the state variable x from (2) and (3), we obtain

$$8u^3 + 27v^2 = 0 \quad (7)$$

Equation (4) is the projection of the three-dimensional manifold of Equation (.4) onto the control space (u-v). The cusp manifold and the bifurcation set are shown in Figure (2) Equation (2) has three real roots within the bifurcation set region, or when:

$$8u^3 + 27v^2 < 0 \quad (8)$$

But when

$$8u^3 + 27v^2 > 0 \quad (9)$$

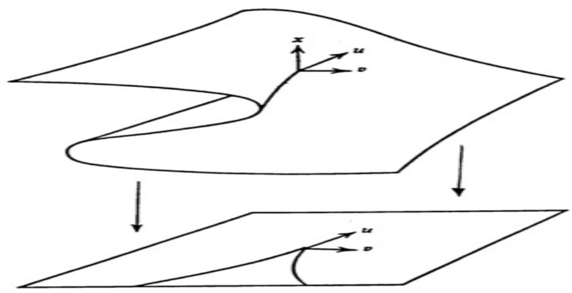

Figure 2: Cusp manifold and its bifurcation set

There is only one real root. Figure (3)shows the bifurcation set of (u - v) plane in which the Functions *v(x)* is sketched for different values of the parameters *u* and *v*.

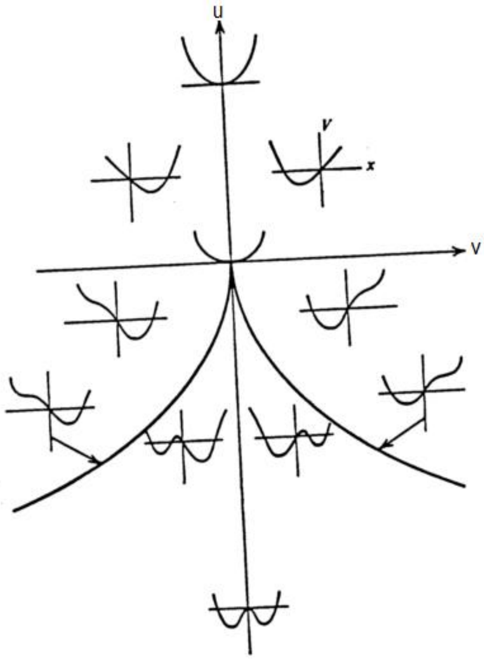

Figure 3: Cusp potential *V(X)* at different values of the control variables

The classification can be extended to codimension 5, in which, case four new functions appear, namely x^7, x^2y-y^5, x^2y+$y^5$and x^3 + y^3 . But for codimension *6* or more the classification becomes infinite (see [9] or [ll]). In applications we can view Thom's theorem as saying the following. The function *f* on its own may be topologically unstable: small perturbations behave differently. The unfolding of *f* a however, captures all the different types of perturbation in a single family. Thus in a physical situation, when we observe *f,*we expect to find the rest of its unfolding too (provided all perturbations are in principle allowed. Symmetry conditions or suchlike can prevent this). Books on bifurcation theory are full of diagrams like figure 3. They should consider figure 4b too since in actual fact this is also going to occur. More: figure (3) is "typical" in a way that 4a is not. And of course figure 2 is what is really relevant. (For a ''perturbation" application of the elliptic umbilic, see a fluid dynamics example due to Michael Berry, reported in [14].).

3. CONCLUSION

In this manuscript, Cusp catastrophes model and the application of singularity theory concepts were used to enumerate the complete bifurcation structures of a two-degree-of-freedom system is believed to be new insofar as the non-linear vibration literature is concerned. Moreover, the approach presented in this paper can deal with some quasiglobal forms of behaviour which cannot be dealt with by various previous methods such as those quoted above.The other main contribution here is the identification of the bifurcation phenomena shown in Figures 3 and 4. The results in Figure 2 show that one can control vibration in non-linear systems by an appropriate choice of system parameters, as suggested by some regions in the hypersurface where the amplitude of the bifurcating solution is always zero.

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