Survey on Transformations for Infinite Series to Continued Fractions with MATLAB Program for Computing Some

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ABSTRACT

In this survey , our aim is to represent to the reader a fascinating and a beautiful approach called Continued Fraction Technique (C.F.T.) to evaluate infinite series by transformation a series to its corresponding equivalent C.F. algorithm. First , We introduce the basic ideas of continued fraction to see how they arise out of high school division and also from solving equations .Second , theorems for Transformation infinite series to their equivalent C.F. are given .Some of these Transformation are to transform alternating series and series having an infinite product of terms.

We observe one can use algebra to simplify the evaluations of the n-th continued fraction term to its equivalent one to generate the $\boldsymbol{\rho}$ - **transformation** easily as in Theorem 2.3. Third , Matlab program software is written to compute some types of series efficiently. We , conclude with a computational remark showing the difficulties in computing some C.F. algorithms due to overflow or underflow for rounding errors , however this will not slamming C.F. for some types of certain problems it can't be avoided specially when evaluating Bessel's functions and its zeros or Hypergeometric series in [4] ,pp.272

Key words : Continued Fraction , Bessel's Function & Matlab Language

1. INTRODUCTION TO CONTINUED FRACTIONS

Here , We introduce the basics ideas of continued fractions to see how they arise out of high school division and also from solving equations

1.1 Continued Fraction Arise from Repeated Division.

Example 1. 1 Take for instance, high school division of 68 into 157 : $\frac{157}{68} = 2 + \frac{21}{68}$. Inverting the fraction $\frac{21}{68}$,we can write $\frac{157}{68}$ as $\frac{157}{68} = 2 + \frac{1}{\frac{68}{21}}$. similarly ,we can further divide $\frac{68}{21} = 3 + \frac{5}{21} = 3 + \frac{1}{\frac{21}{5}}$. Now we write $\frac{157}{68}$ in somewhat fancy way as : $\frac{157}{68} = 2 + \frac{1}{3+\frac{1}{\frac{21}{5}}}$ and finally as

$$\frac{157}{68} = 2 + \frac{1}{3+\frac{1}{4+\frac{1}{5}}} \qquad (1)$$

Since, 5 is now a whole number , our repeated division stops.

The expression on the right in (1) is called a finite simple continued fraction. There are many ways to express the right-hand side , but we shall stick with the following two notations:

$$\langle 2;3,4,5\rangle \quad or \quad 2+\frac{1}{3+}\frac{1}{4+}\frac{1}{5}$$

$$represent \quad 2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5}}}$$

This implies continued fractions by repeated divisions continuously, we can express any two integers a and b with a $\neq 0$ as b/a as a finite simple continued fractions.

1.2 Continued Fraction arise from when Solving Equation

When we are trying to solve equations continued fraction arise as in following example 1.2

Example 1.2 .

Suppose we want to find the positive solution x to the equation

$$x^2-x-2=0$$

. Notice that two 2 is only the positive solution. On the other hand writing $x^2-x-2=0$ as

x^2-x+2 and dividing by x ,we get $x=1+\frac{2}{x}$ $or\ since\ x=2$, $2=1+\frac{2}{x}$ Now if we replace the denominator with $2=1+\frac{2}{x}$ to get $2=1+\frac{2}{1+\frac{2}{x}}$

Repeating this many times , we can write

$$2=1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{\cdots}{1+\cfrac{2}{x}}}}}$$

Repeating this to "" to infinity "" , we write

$$2=1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1+\ddots}}}}$$

This means that any whole number can be written in such a way as an infinite continued fraction as we will be define next.

1.3 Basic Definitions for Continued Fraction (C.F.)

If no divisions by zero are allowed and $a_k's$ & $b_k's$ are real numbers ,then

general continued fraction is written as

$$(2) \quad b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\cfrac{\ddots}{b_{n-1}+\cfrac{a_n}{b_n}}}}}$$

If $a_m=0$ for some m ,then

$$(3)\ b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\cfrac{\ddots}{b_{n-1}+\cfrac{a_n}{b_n}}}}}=b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\cfrac{\ddots}{b_{m-2}+\cfrac{a_{m-1}}{b_{m-1}}}}}}$$

The continued fraction is called SIMPLE if all the $a_k's\ are\ 1$ and the $b_k's$ are integers with b_k positive for each $k\geq 1.$ Instead of writing the continued fraction as above which takes a lot of spaces ,we shorten it to :

$$b_0+\frac{a_1}{b_1+}\frac{a_2}{b_1+}\dots\dots\frac{a_n}{b_n}\dots\dots etc$$

In simple fraction case when a's are ones

we shorten the notation to

$$b_0 + \frac{a_1}{b_1} \frac{a_2}{b_1} \dots \dots \frac{a_n}{b_n} = \langle b_0; b_1, b_2, b_3, \dots, b_n \rangle$$

$$or = \langle 0; b_1, b_2, b_3, \dots, b_n \rangle \quad (4)$$

1.4 THE n-th Convergent C_N of the Infinite Continued Fraction (C.F.)

We now discuss infinite continued fractions.
Let $\{a_n\}, n = 1,2,..$ *and* $\{b_n\}, n = 0,1,2,3, \dots$ *be sequences of real numbers and suppose that*

$$C_n := b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_1+} \dots \dots \frac{a_n}{b_n} \qquad (5)$$

is defined for all n. We call C_n the n-th convergent of the continued fraction . If the limit , limit C_n ,exists then we say the infinite continued fraction

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{\ddots}{b_{n-1} + \cfrac{a_n}{b_n + \ddots}}}}} \qquad (6)$$

$$\text{or} \quad b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_1+} \dots \frac{a_n}{b_n} \dots$$

Converges and we use either of these notations to denote the limiting value limit C_n .

If C_n is a simple continued fraction , then for the limit we use the notation (4) as

$$\langle b_0; b_1, b_2, b_3, \dots, b_n \dots etc \rangle$$

$$\lim_{n\to\infty} \langle b_0; b_1, b_2, b_3, \dots, b_n \rangle = \lim_{n\to\infty} C_n \quad (5)$$

2. THOERIES TO TRANSFORM ININITE SERIES.

Here ,we state three important theories to transform an infinite series to its corresponding equivalent continued fraction . If the limit of continued fraction exits then it must be equal to its sum whenever the series is convergent. Since the convergent of infinite continued fraction is so important we will list a theorem to verify the convergence. The proof of these transformation is not difficult provided that one must use the induction rule.

2.1 Theorem 2.1 ((Transform an Alternating Series))

If $\alpha_1, \alpha_2, \alpha_3, ..$ are nonzero real number with $\alpha_k \neq \alpha_{k-1}$ for all k then for any $n \in \mathbb{N}$

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{\alpha_k} = \cfrac{1}{\alpha_1 + \cfrac{\alpha_1^2}{\alpha_2 - \alpha_1 + \cfrac{\alpha_2^2}{\alpha_3 - \alpha_2 + \cfrac{\ddots}{+\cfrac{\alpha_{n-1}^2}{\alpha_n - \alpha_{n-1}}}}}}$$

In particular ,taking $n \to \infty$,we conclude that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\alpha_k} = \frac{1}{\alpha_1 |} \frac{\alpha_1^2}{\alpha_2 - \alpha_1 |} \frac{\alpha_2^2}{\alpha_3 - \alpha_2 |} \frac{\alpha_3^2}{\alpha_4 - \alpha_3 |} \dots \frac{\alpha_{n-1}^2}{\alpha_n - \alpha_{n-1} |} \dots$$

EXAMPLE 2.1 Transform the series

$$\ln(2) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots \textit{to C.F.}$$

Setting

$\alpha_k = k$,we *get* two beautifully expressions

$$\ln(2) - \frac{1}{1+} \frac{1^2}{1+} \frac{2^2}{1+} \frac{3^2}{1+} \dots$$

$$\ln(2) = \cfrac{1}{1 + \cfrac{1^2}{1 + \cfrac{2^2}{1 + \cfrac{3^2}{1 + \cfrac{4^2}{1 + \ddots}}}}}$$

Example 2.2 Transform the series for π into continued fraction

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \dots$$

Setting

$\alpha_k = 2k-1$,we *get* two beautifully expressions

$$(7) \qquad \frac{\pi}{4} = \frac{1}{1+}\frac{1^2}{2+}\frac{3^2}{2+}\frac{5^2}{2+}\frac{7^2}{2+}\cdots\cdots$$

Now , if we invert (7) we obtain the incredible expansion

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ddots}}}}$$

2.2 Theorem 2.2 Transform Series having Infinite Products Terms to C.F.

For any real sequences

$\alpha_1, \alpha_2, \alpha_3, ..$ with $\alpha_k \neq 0, 1$,we have

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{\alpha_1\alpha_2\dots\alpha_k} = \cfrac{1}{\alpha_1+\cfrac{\alpha_1}{\alpha_2-1+\cfrac{\alpha_2}{\alpha_3-1+\cfrac{\ddots}{\alpha_n-1+\cfrac{\alpha_{n-1}}{\alpha_n}}}}}$$

In particular ,taking $n\to\infty$,we conclude that

$$(8) \qquad \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\alpha_1\alpha_2\dots\alpha_k} = \frac{1}{\alpha_1+}\frac{\alpha_1}{\alpha_2-1+}\frac{\alpha_2}{\alpha_3-1+}\frac{\alpha_3}{\alpha_4-1+}\cdots\frac{\alpha_{n-1}}{\alpha_n-1+}\cdots$$

Example 2.3 Continued Fraction for e

From the series

$$e^{-x} = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{k-1}}{(k-1)!} = \frac{1}{1}-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}\dots\dots \textit{and formula } (6)$$

if we put $\alpha_n = \frac{n!}{x^n}$ into the n-th fraction $\frac{\alpha_{n-1}{}^2}{\alpha_n-\alpha_{n-1}}$,

$$\textit{we get } \frac{\alpha_{n-1}{}^2}{\alpha_n-\alpha_{n-1}+} = \frac{(n-1)!\,x}{n-x} \quad \text{with } \rho_n = x^n \quad \textit{as from theorem } 2.3$$

$$(9) \qquad e^{-x} = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{k-1}}{k-1!} = \frac{1}{1+}\frac{x}{2-x+}\frac{2x}{3-x+}\frac{6x}{4-x+}\frac{24x}{5-x+}\dots\dots$$

From the series

$$e^{-x} - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{k-1}}{(k-1)!} = \frac{1}{1}-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}\dots\dots \textit{and formula } (8)$$

if we put $\alpha_k = \frac{k}{x}$ into the n-th fraction $\frac{\alpha_{n-1}}{\alpha_n-1}$,

$$\textit{we get } \frac{\alpha_{n-1}}{\alpha_n-1} - \frac{(n-1)}{n-x} \text{ with } \rho_n - x^n \quad \textit{as from theorem } 2.3 \quad \textit{and}$$

$$(10) \qquad e^{-x} = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{k-1}}{(k-1)!} = \frac{1}{1+}\frac{1}{2-x+}\frac{2}{3-x+}\frac{3}{4-x+}\frac{4}{5-x+}\dots\dots$$

from (10) if we put x=1 and invert it , we find

$$e-1 = 1 + \cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\ddots}}}}$$

We observe that from formulas (9) & (10) according to theorem one uses can obtained two different associative series for the same function for which rapid convergence sometimes can't be achieved. Second, in the n-th fraction we substitute the n-th α_n term directly observing and looking to reasonable transformations such as $\rho_1, \rho_1\rho_2, \rho_1\rho_2\rho_3, \cdots$ as in the following Theorem 2.3.

2.3 Theorem 2.3 Transform Continued Fraction into Equivalent One.

For an y real numbers

$a_1, a_2, a_3, \dots\dots$ *and* $b_1, b_2, b_3, \dots\dots$ and nonzero constants $, \rho_1, \rho_2, \rho_3, \dots$

we have

$$(11) \qquad b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_1+}\dots\dots\frac{a_n}{b_n} = b_0 + \frac{\rho_1 a_1}{\rho_1 b_1+}\frac{\rho_1\rho_2 a_2}{\rho_2 b_1+}\dots\dots\frac{\rho_{n-1}\rho_n a_n}{\rho_n b_n}$$

Example 2.4 Continued Fraction for arctan(x)

$$\arctan(x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k-1}}{2k-1} = \frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}\dots$$

Now if we put $\propto_n = \frac{2n-1}{x^{2n-1}}$ into the n-th fraction $\frac{\propto_{n-1}{}^2}{\propto_n - \propto_{n-1}}$ *into formula* (6) ,

$$we\ get\ \frac{\propto_{n-1}{}^2}{\propto_n - \propto_{n-1}} = \frac{\left(\frac{2n-3}{x^{2n-3}}\right)^2}{\frac{2n-1}{x^{2n-1}} - \frac{2n-3}{x^{2n-3}}} = \frac{(2n-3)^2 x^2}{(2n-1)-(2n-3)x^2}$$

, $\rho_n = (2n-3)x^2$,n ≥ 1 and

$$\arctan(x) = \frac{x}{1+} \frac{1^2 x^2}{3-x^2+} \frac{3^2 x^2}{5-3x^2+} \frac{5^2 x^2}{7-5x^2+} \cdots$$

or in a fancier approach

$$(12)\quad \arctan(x) = \cfrac{x}{1+\cfrac{x^2}{(3-x^2)+\cfrac{3^2x^2}{(5-3x^2)+\cfrac{5^2x^2}{(7-5x^2)+\cfrac{7^2x^2}{(9-7x^2)+\ddots}}}}}$$

Alternatively ,if we apply theorem 2.2 first , and setting

$$\propto_1 = \frac{1}{x}, \propto_2 = \frac{3}{x^2}, \propto_3 = \frac{5}{3x^2}, \propto_4 = \frac{7}{5x^2}, \ldots, \qquad \propto_n = \frac{2n-1}{(2n-3)x^2} \text{ for } n \geq 2$$

into formula (8)

$$\frac{1}{\propto_1} - \frac{1}{\propto_1\propto_2} + \frac{1}{\propto_1\propto_2\propto_3} + \cdots .. = \frac{1}{\propto_1 +} \frac{\propto_1}{\propto_2 - 1 +} \frac{\propto_2}{\propto_3 - 1 +} \frac{\propto_3}{\propto_4 - 1 +} \cdots \frac{\propto_{n-1}}{\propto_n - 1 +}$$

We obtain (13)

$$\arctan(x) = \frac{1}{\frac{1}{x}+} \frac{\frac{1}{x}}{\frac{3}{x^2}-1+} \frac{\frac{3}{x^2}}{\frac{5}{3x^2}-1+} \frac{\frac{5}{3x^2}}{\frac{7}{5x^2}-1+} \cdots \frac{\frac{2n-1}{(2n-3)x^2}}{+\frac{2n+1}{(2n-1)x^2}-1+}$$

Now ,setting

$$\rho_1 = x\ , \rho_2 = x^2, \rho_3 = 3x^2, \rho_4 = 5x^2 \text{ and generally } \rho_n = (2n-3)x^2\ ,\ n \geq 1$$

in formula (13) for $\arctan(x)$ we will obtain the expression (12) for arctan(x) .

Here , we recommend those who are strong in algebra shall dive into the n-th fraction term to simplify it to get the transformation easily

$$\frac{\frac{2n-1}{(2n-3)x^2}}{\frac{2n+1}{(2n-1)x^2}-1} = \frac{\propto_n}{\propto_{n+1} - 1}.$$

If we put x=1 into formula (12) and inverting ,**we get Lord Brounckers's formula** :

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ddots}}}}$$

2.4 More Transformation Identites for Series to C.F.

Theorem 2.4 ((TRANSTORM A SERIES OF POSITIVE TERMS))

If $\propto_1, \propto_2, \propto_3, ..$ are nonzero real number with $\propto_k \neq \propto_{k-1}$ for all k , then for any n $\in \mathbb{N}$

$$\sum_{k=1}^{n} \frac{1}{\propto_k} = \cfrac{1}{\propto_1 - \cfrac{\propto_1{}^2}{\propto_2 + \propto_1 - \cfrac{\propto_2{}^2}{\propto_3 + \propto_2 - \cfrac{\ddots}{\propto_n + \propto_{n-1} - \cfrac{\propto_{n-1}{}^2}{\propto_n + \propto_{n-1}}}}}}$$

In particular ,taking $n \to \infty$,we conclude that

$$(14)\quad \sum_{k=1}^{\infty} \frac{1}{\propto_k} = \frac{1}{\propto_1 -} \frac{\propto_1{}^2}{\propto_2 + \propto_1 -} \frac{\propto_2{}^2}{\propto_3 + \propto_2 -} \frac{\propto_3{}^2}{\propto_4 + \propto_3 -} \cdots \frac{\propto_{n-1}{}^2}{\propto_n + \propto_{n-1} -} \cdots$$

Example 2.5 Continued Fraction for Euler's sum $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots ..$

Apply formula 14) with $\propto_n = n^2$

$$(15)\quad \frac{\pi^2}{6} = \frac{1}{1^2 -} \frac{1^4}{2^2 + 1^2 -} \frac{2^4}{3^2 + 2^2 -} \frac{3^4}{4^2 + 3^2 -} \cdots \frac{(n-1)^4}{(n)^2 + (n-1)^2 -} \cdots$$

Now if we invert the formula (15) , we get the splendid one as

$$\frac{6}{\pi^2} - 0^2 + 1^2 - \cfrac{1^4}{1^2 + 2^2 - \cfrac{2^4}{2^2 + 3^2 - \cfrac{3^4}{3^2 + 4^2 - \cfrac{4^4}{4^2 + 5^2 - \ddots}}}}$$

Theorem 2.5 ((TRANSTORM AN ALTERNATING SERIES OF TERMS AS PRODUCTS OF TWO ELEMENTS))

(16) *If the series* $S = \frac{1}{\alpha_1\alpha_2} - \frac{1}{\alpha_2\alpha_3} + \frac{1}{\alpha_3\alpha_4} - \frac{1}{\alpha_4\alpha_5} + \frac{1}{\alpha_5\alpha_6} - \cdots$

Then its C.F. is

$$(17)\quad \frac{1}{\alpha_1 S} = \alpha_2 + \cfrac{\alpha_1\alpha_2}{\alpha_3 - \alpha_1 + \cfrac{\alpha_2\alpha_3}{\alpha_4 - \alpha_2 + \cfrac{\alpha_3\alpha_4}{\alpha_5 - \alpha_3 + \cfrac{\alpha_4\alpha_5}{\alpha_6 - \alpha_4 + etc}}}}$$

Example 2.6 To adapt this form in (17) , consider

$$\ln(2) - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots$$

$$\text{and} \qquad \ln(2) - 1 == -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots$$

adding these two equations

We get

$$2\ln(2) - 1 = \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + \frac{1}{5.6} - etc \ldots$$

Therefore, the sum $S = 2\ln(2) \quad -1$

Now let

$$\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \alpha_4 = 4, \alpha_5 = 5, \ldots.$$

Therefore the equivalent C.F. is

$$\frac{1}{2\ln(2) - 1} = 2 + \cfrac{1.2}{2 + \cfrac{2.3}{2 + \cfrac{3.4}{2 + \cfrac{4.5}{2 + etc}}}}$$

Similarly, we know

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} - \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots$$

and

$$\frac{\pi}{4} - 1 = -\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots \quad \textit{adding these to get}$$

$$\frac{\pi}{2} - \frac{1}{2} - \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} + \frac{2}{7.9} - \cdots$$

Then C.F. arise as

$$\frac{4}{\pi - 2} = 3 + \cfrac{1.3}{4 + \cfrac{3.5}{4 + \cfrac{5.7}{4 + \cfrac{7.9}{4 + etc}}}}$$

Theorem 2.6 ((TRANSTORM AN INFINITE PRODUCTS TO C.F.))

If $\alpha_1, \alpha_2, \alpha_3, \ldots$ are nonzero real number with $\alpha_k \neq \alpha_{k-1}$ and $\alpha_k \neq 0, -1$ for all k.

Define sequences by

$a_1, a_2, a_3, \ldots$ *and* $b_1, b_2, b_3, \ldots$ by $a_1 = (1+\alpha_0)\alpha_1$, $b_0 = (1+\alpha_0)$

$b_1 = 1$ *and* $a_n = -(1+\alpha_{n-1})\frac{\alpha_n}{\alpha_{n-1}}$, $b_n = 1 - a_n$ for $n = 2, 3, 4, \ldots$

Prove (say by induction) that for any $n \in \mathbb{N}$

$$(18)\quad \prod_{k=0}^{n}(1+\alpha_k) = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_1 +} \cdots \frac{a_n}{b_n}$$

Taking $n \to \infty$,we get a similar formula for infinite products and fractions

Example 2.7 To adapt this form in (18) , consider

$$\frac{\sin(\pi x)}{\pi x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right) = (1-x)(1+x)\left(1-\frac{x}{2}\right)\left(1+\frac{x}{2}\right)\left(1-\frac{x}{3}\right)\left(1+\frac{x}{3}\right)\ldots$$

,then we can derive the continued fraction expansion

$$(19)\quad \frac{\sin(\pi x)}{\pi x} = 1 - \frac{x}{1+} \frac{1.(1-x)}{x+} \frac{1.(1+x)}{1-x+} \frac{2.(2-x)}{x+} \frac{2.(2+x)}{1-x+} \ldots$$

Now , if one put

$x = \frac{1}{2}$ *, we get formula* (20)

(20)

$$\frac{\pi}{2} = 1 + \cfrac{1}{1 + \cfrac{1.3}{1 + \cfrac{2.3}{1 + \cfrac{2.5}{1 + etc}}}}$$

But if we put

$x = -\frac{1}{2}$ *, we get formula* (21)

(21)

$$\frac{2}{\pi} = 1 + \cfrac{1}{2 - \cfrac{3}{1 + \cfrac{2}{3 - \cfrac{10}{1 + etc}}}}$$

Theorem 2.7 ((TRANSTORM AN INFINITE SERIES TO C.F.))

If

$\alpha_1, \alpha_2, \alpha_3, ..$ are nonzero real number with $\alpha_k \neq \alpha_{k-1}$ for all k, and the infinite

series $\sum_{k=0}^{\infty} \alpha_k x^k$ *converges ,then it can be shown that itsequivalent C.F. is*

$$(22)\quad \sum_{k=0}^{\infty} \alpha_k x^k = \alpha_0 + \frac{\alpha_1 x}{1+} \frac{-\alpha_2 x}{\alpha_1 + \alpha_2 x +} \frac{-\alpha_1 \alpha_3 x}{\alpha_2 + \alpha_3 x +} \cdots \frac{-\alpha_{n-2} \alpha_n x}{\alpha_{n-1} + \alpha_n x +} \cdots$$

EXAMPLE 2.7 To adapt this form in (.22) , consider

$$\arctan(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{2k-1} = x\left(1 - \frac{y^1}{3} + \frac{y^2}{5} - \frac{y^3}{7} + \frac{y^4}{9} \ldots\right) \text{ with } y = x^2$$

Now applying the formula (22) on $\left(1 - \frac{y^1}{3} + \frac{y^2}{5} - \frac{y^3}{7} + \frac{y^4}{9} \ldots\right)$, we get (12)

3. CONVERGENCE AND RECURRENCE RELATIONS

3.1 Walli's Recurrence Relations

Let us call a continued fraction

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_1 +} \cdots \cdots \frac{a_n}{b_n} \cdots \qquad (23)$$

nonnegative if the

a_n' and b_r' s are real numbers with $b_n > 0, a_n \geq 0$ for all $n \geq 1$

.

Let $\{a_n\}, n = 1,2, \ldots$ *and* $\{b_n\}, n = 0,1,2, \ldots$ *be sequences of real numbers*

with $b_n > 0 , a_n \geq 0$ for all $n \geq 1$,

then the Wallis-Euler recurrence relations for

the n-th convergent continued fraction C_n defined eariler by Eqn(5) is given by

$C_n := \frac{P_n}{Q_n}$ *, where the sequenses* $\{P_n\}$ *and* $\{Q_n\}$ *are given by* the Wallis – Euler recurrence in the formula (1.24)

$$p_n = b_n p_{n-1} + a_n p_{n-2} \quad , \quad q_n = b_n q_{n-1} + a_n q_{n-2} ,$$

$$p_{-1} = 1 , p_0 = b_0 , q_{-1} = 0 , q_0 = 1 \quad (24)$$

n=0,1 ,2 , 3 ,………

Now , it is clear that

$$p_1 = b_1 p_0 + a_1 p_{-1} = b_1 b_0 + a_1 \ , \ q_1 = b_1 q_0 + a_1 q_{-1} = b_1$$

Further , it can be shown that

$q_n > 0$, *for all* $n = 0, 1, 2, \ldots$

3.2 Fundamental Recurrence Relations

for all $n \geq 1$, the following identities hold

$$P_n q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} a_1, a_2, a_3, \ldots, a_n$$

$$P_n q_{n-2} - P_{n-2} Q_n = (-1)^n b_n \ a_1, a_2, a_3, \ldots, a_n \quad (25)$$

and where the formula for $C_n - C_{n-1}$ is only valid for $n \geq 2$

$$C_n - C_{n-1} = \frac{(-1)^{n-1} a_1, a_2, a_3, \ldots, a_n}{Q_n Q_{n-1}}$$

$$C_n - C_{n-2} = \frac{(-1)^n b_n \ a_1, a_2, a_3, \ldots, a_n}{Q_n Q_{n-2}} \quad (26)$$

3.3 Simple Fundamental Recurrence Relations.

For simple continued fractions, for all $n \geq 1$, if formula (24) holds then $C_n := \frac{P_n}{Q_n}$

for all $n \geq 0$ *and for any* $x > 0$,

$$(27)\quad (b_0; b_1, b_2, b_3, \ldots, b_n \ldots etc) = \frac{x P_{n-1} + P_{n-2}}{x q_{n-1} + q_{n-2}} \quad , n = 1, 2, 3, \ldots$$

Moreover , for all

$n \geq 1$, the following identities hold

$$P_n q_{n-1} - P_{n-1} Q_n = (-1)^{n-1}$$

$$P_n q_{n-2} - P_{n-2} Q_n = (-1)^n b_n$$

and

$$C_n - C_{n-1} = \frac{(-1)^{n-1}}{Q_n Q_{n-1}} \quad , \quad C_n - C_{n-1} = \frac{(-1)^n b_n}{Q_n Q_{n-2}}$$

where

$C_n - C_{n-1}$ is only valid for $n \geq 2$

3.4 Convergence Results for C.F.

It can be shown that limit C_n exists if and only if

$$(28)\quad C_{2n} \quad C_{2n-1} = \frac{-a_1 a_2 a_3 \ldots a_n}{Q_{2n} Q_{2n-1}} \to 0 \text{ as } n \to \infty$$

The formula (28) holds if the following one **condition theorem 3.1 is satisfied**

Theorem 3.1 Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences such that

$$(29)\quad a_n, b_n > 0 \text{ for } n \ge 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{b_n b_{n+1}}{a_{n+1}} = \infty$$

Then formula (28) holds and so the C.F.

$$:= \; b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_1+} \cdots \frac{a_n}{b_n+} \,.\, converges$$

Moreover , for any even j and odd k , we have

$$c_0 < c_2 < c_4, \ldots . < c_j < \cdots < \square < \cdots < c_k \ldots \ldots . . < c_5 < c_3 < c_1$$

4. MATLAB PROGRAM TO COMPUTE CONTINUED FRACTIONS

When the series or functions such as arctan(x) are expressed in C.F. we advice usually one to investigate where the series is convergent or not ,even if the series has already been transformed in its equivalent C.F. If there is no idea about the series but an expression of a continued fraction required to be evaluated, then it must be examined by theorem 3.1 for its convergence. Once that is valid our MATLAB program can be used to compute the C.F. using Walli's Euler recurrence formulae (24) to evaluate it. In all the following Matlab program in the C.F. the a's and b's for nominator and denominator are replaced by f's and g's respectively .Lastly , these programs compute a C.F. series for which the denominators are being zeros or can produce an overflow or underflow for either nominator and denominator**.** The Matlab Programs are in Fiq.(1),…,Fiq.(4) **in the APPENDIX**

Example 4.1 The Matlab program in Fiq.(1) compute exp(1)-2 as from C.F. formula (10) using backwards recurrence

Example 4.2 The Matlab program in Fiq.(2) compute $\pi - 3$ from the C.F. given by

Lange [4] as

$$\pi = 3 + \frac{1}{6+} \frac{9}{6+} \frac{25}{6+} \frac{49}{6+} \frac{81}{6+} \frac{121}{6+} \ldots . (30) \; using \;\; \text{forward recurrence}$$

The idea of **the program forrec1.m** is known when two consecutive n-th convergent continued fraction

$C_n := \dfrac{P_n}{Q_n}$ given by Eqn. (5) acheive the reguired tolerance accuracy tau . But **the program forrec2.m in Fiq.(3)** compute $\pi - 3$ *using* forward recurrence *with rescaling check after every 10 steps*

Example 4.3 The Matlab program in Fiq.(4) compute $\pi - 3$ from the C.F. given by Lange [4] as

$$\pi = 3 + \frac{1}{6+} \frac{9}{6+} \frac{25}{6+} \frac{49}{5+} \frac{81}{6+} \frac{121}{6+} \ldots . (30) \; using \;\; \text{forward recurrence}$$

with newly adjusted derivation in [20] with observation that when b_0 =0 in the well known Walli's recurrence formula (24) with P's and Q's are replaced by A's and B's respectively in (30)

$$r_k = \frac{B_k}{A_k} = \frac{g_k r_{k-1} + f_k v_{k-1}}{g_k + f_k u_{k-1}} \;, u_k = \frac{A_{k-1}}{A_k} = \frac{1}{g_{k} + f_k u_{k-1}} \,,$$

$$v_k = \frac{B_{k-1}}{A_k} = \frac{r_{k-1}}{g_{k} + f_k u_{k-1}} \qquad (30)$$

with the initial values as in Table I

Table I initial values for c.f. $= \dfrac{f_1}{g_1} \dfrac{f_2}{g_2} \dfrac{f_3}{g_3} \ldots$ ***computed by (30)***

K	0	1
A_k	0	f_1
B_k	1	g_1
$r_k = \frac{B_k}{A_k}$		$\frac{g_1}{f_1}$
$u_k = \frac{A_{k-1}}{A_k}$		0
$v_k = \frac{B_{k-1}}{A_k}$		$\frac{1}{f_1}$

5. COMPUTATIONAL REMARK

We have observe that expressing a particular series such as the series for natural exponential exp(x) to continued fraction expansion can have many different form of expansion as can be seen [18]. Some of these may lead to difficulty in evaluation and computing their sum .Others can be transformed to its equivalent C.F. using the $\rho - transform$ formula as in Eqn.(11).

In Fiq.(4) Program forwrec3.m to compute $\pi - 3$, the number of terms is at least 500 . This indicates that there are so many efficient techniques to sum the series as in [17] and [18] using few number of terms. However , C.F. technique are very important to find the zeros of Bessel's function of the first kind or hypergeometric function as in [19] and [4] respectively .

Further , writing one his own program in Matlab or Mathematica can achieved great benefit in complexity or tracing the program easily instead of MuPad interface or Mathematica built-in functions.

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APPENDIX

```
function  y=cf(f,g)
% CF evaluates a continued
% fraction using
%  backwards recurrence
    disp( ['     ' 'f'  '     ' 'g' ]) ;
    % check coefficient of C.F.
  disp([ f' g'])    s=0;
   for k= length(f):-1:1
       s=f(k)/(s+g(k));
   end
```

```
>>   m=8; n=2*m;

>>   for k=1:n

         g(k)=2 +k-1 ;  f(k)=g(k);

     end

>>  format long
```

```
    f     g
    2     2
    3     3
    4     4
    .     .
    .     .
    16    16
    17    17
  y = 0.718281828459045
```

Fiq.(1) Program function cf.m to compute exp(1)-2 with its COMMAND.

```
function [cf , k] = forwrec1(ff , gg , kmax , r , tau)
% FORWREC1 evaluates the continued fraction given
%  by  the two vectors ff and gg by forward recurrence
%       every r step it is checked if two consecutive
%       partial fractions match to tolerance tau
p=1 ; q=2; n1 = 2;
a(n1+1)= 0 ; b(n1+1) = 1;
a(n1-1)= ff(1) ; b(n1-1) = gg(1);
for k=2:kmax
    f=ff(k) ; g=gg(k);
    a(n1+p) = g*a(n1-p) + f*a(n1+p);
    b(n1+p) = g*b(n1-p) + f*b(n1+p);
    if k==q*r | k==kmax
        cf = a(n1+p) / b(n1+p) ;
        cg = a(n1-p) / b(n1-p) ;
        if abs(cf-cg ) < tau
            break
        end
        q=q+1
    end
    p=-p ;
```

Fiq.(2) Program forwrec1.m with its command to evaluate $\pi - 3$

```
n=100;
f=1:n ; f=ones(size(f)) + 2*f ; f = [1 f.*f];
g=6*ones(size(f));
format long

  y=cf(f,g)
  y = 0.141592889142081
[yy ,k] = forwrec1(f ,g , n+1 , 3 , 1e-10)
    yy  =  0.141592889142081
```



```
function [cf , k] = forwrec2(ff , gg , kmax  , tau)
% FORWREC2 evaluates the continued fraction
%  given by ff and gg the two vectors by
%    forward recurrence convergence and
%     rescaling are checked for every
%      10 steps
p=1 ;  n1 = 2;
a(n1+1)= 0 ; b(n1+1) = 1;
a(n1-1)= ff(1) ; b(n1-1) = gg(1);
for k=0:10:kmax-11
 for j=2:11
   f=ff(j+k) ; g=gg(j+k);
   a(n1+p) = g*a(n1-p) + f*a(n1+p);
   b(n1+p) = g*b(n1-p) + f*b(n1+p);
   p=-p;
  end
      cf = a(n1+p) / b(n1+p) ;
      cg = a(n1-p) / b(n1-p) ;
      if abs(cf-cg ) < tau
         break
      end
   maxx=abs(a(n1+1))+abs(a(n1-1))
               +abs(b(n1+1))+abs(b(n1-1));
   if maxx>1e20
       d=1e-20;
   elseif maxx<1e-20
       d=1e20 ;
   else
       d=1;
   end;
    a(n1+p) = d*a(n1+p) ;   b(n1+p) = d*b(n1+p);
    a(n1-p) = d*a(n1-p) ;   b(n1-p) = d*b(n1-p);
end
```

Fiq.(3) Program forwrec2.m to compute π 3

```
>>  n=100;
>>  f=1:n ; f=ones(size(f))+2*f ;
>>  f=[1  f.*f];
>>  g=6*ones(size(f));
>>  format long

>>  y=cf(f,g)
    y =    0.141592656124989

>>  pi-3  = 0.141592653589793

>>  [yy ,k] =forwrec2(f,g,n+1 ,0)
    yy = 0.141592651038063
    k = 450
```

```
function [y , k] = forwrec3(f , g  , kmax ,  tau)
% FORWREC3  computes successive convergents
%  for  the continued fraction given by the two
%    vectors nomonators  f and  denominators g .

 r(1)= g(1) / f(1) ;
 u(1) =  0 ;
  v(1) = 1 /f(1) ;
  for k=2:kmax
     u(k) = 1 / (g(k) + f(k)*u(k-1));
      r(k) = (g(k)*r(k-1) + f(k)*v(k-1))*u(k) ;
      v(k) = r(k-1) * u(k);
       if abs(1/r(k) - 1/r(k-1) ) < tau
           break
      end

  end
   y = 1/r(k);

  %  Type in the Command Window
  %  for n=5000 : 5000 : 15000
  %  f=1:n ; f=ones(size(f))+2*f ; f = [1 f.*f ];
  %  g = 6*ones(size(f));
  %  [y  k] = forwrec3(f,g,n+1 ,0)
  %  Error= abs(pi-3-y);
  %  disp(Error)
  %  end
```

Fiq.(4) Program forwrec3.m to compute $\pi - 3$

```
>>  format long
 >>  for n=5000 : 5000 : 15000
       f=1:n ; f=ones(size(f))+2*f ; f = [1 f.*f ];
       g = 6*ones(size(f));
       [y  k] = forwrec3(f,g,n+1 ,0)
       Error= abs(pi-3-y);
        disp(Error)
     end
```

n	$y = A_n/B_n$	Error = abs(pi-3- y)
5001	0.141592653591791	1.997763066086122e-12
10001	0.141592653590042	2.489675132721914e-13
15001	0.141592653589864	7.105427357601002e-14